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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equations over the real numbers.

a) $\frac{x+3}{3} = \frac{x^2+3x-4}{2x^2+4x-16} \cdot (x+3)$

b) $\lg(6^x - 96) - 2 = \lg 2 + \lg 6$

(11 points)

Solution: a) We factor the quadratic expressions:

$$x^2 + 3x - 4 = (x+4)(x-1), \quad 2x^2 + 4x - 16 = 2(x-2)(x+4)$$

Conditions: $x \neq 2$ and $x \neq -4$. Since $x \neq -4$, we can simplify by $x+4$, and multiplying by $6(x-2)$ we get $2(x+3)(x-2) = 3(x-1)(x+3)$, that is, $(x+3)(x-1) = 0$. The solutions are $x = -3$ and $x = -1$; both satisfy the conditions.

b) Because of the logarithm, $x > \log_6 96 \approx 2.547$. By the rules of logarithms, and since the logarithm function is strictly increasing, the equation is

$$\frac{6^x - 96}{100} = 12$$

so $6^x = 1296$, $x = 4$. This satisfies the condition, so it is the solution.

2. On a subway escalator, a bag gets down to the subway level in a seconds. A passenger walks the same way on the escalator when it is not working in b seconds. How long does it take the passenger to get down to the subway level on the working escalator if he walks with the same steps as when it is not working?

(12 points)

Solution: Let the length of the way be s . The speed of the escalator is $v_e = \frac{s}{a}$, and the speed of the passenger is $v_p = \frac{s}{b}$. If the passenger walks on the working escalator, the time is

$$\frac{s}{v_e + v_p} = \frac{s}{\frac{s}{a} + \frac{s}{b}} = \frac{ab}{a+b}$$

So the passenger gets down in $\frac{ab}{a+b}$ seconds.

3. Solve the following system of equations over the real numbers.

$$x + y = \frac{1}{3}, \quad \cos \pi x \cdot \cos \pi y = \frac{1}{2}$$

(14 points)

Solution: Multiplying the first equation by π and taking the cosine of both sides:

$$\cos(\pi x + \pi y) = \cos \frac{\pi}{3} = \frac{1}{2}$$

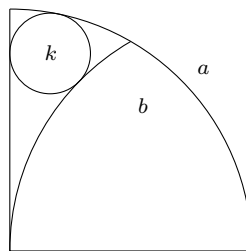
By the addition formula, $\cos \pi x \cdot \cos \pi y - \sin \pi x \cdot \sin \pi y = \frac{1}{2}$, and using the second equation, $\sin \pi x \cdot \sin \pi y = 0$.

From $\sin \pi x = 0$: $\pi x = k\pi$, so $x = k$, $y = \frac{1}{3} - k$, $k \in \mathbb{Z}$.

From $\sin \pi y = 0$: $\pi y = n\pi$, so $y = n$, $x = \frac{1}{3} - n$, $n \in \mathbb{Z}$.

These real numbers are indeed solutions of the system: for example, $\cos k\pi \cdot \cos\left(\frac{\pi}{3} - k\pi\right) = (-1)^k \cdot (-1)^k \cos \frac{\pi}{3} = \frac{1}{2}$.

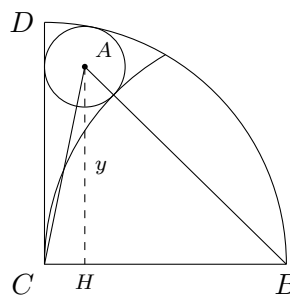
4. From one endpoint of the arc of a quarter circle of radius r (the arc a in the figure), as center, we draw an arc also of radius r (the arc b), which divides the circular sector determined by the quarter circle into two parts.



Find the radius of the circle k inscribed in the smaller part.

(14 points)

Solution: Let C be the center of the quarter circle, B the endpoint of the arc a that is the center of the arc b , and D the other endpoint; so $CB = CD = r$. Let A be the center and x the radius of the circle k , and let H be the projection of A onto CB .



The circle k touches CD , so $CH = x$; it touches the arc a from inside, so $AC = r - x$; and it touches the arc b from outside, so $AB = r + x$. Then $HB = r - x$. Let $y = AH$.

In the triangle ACH : $y^2 = (r - x)^2 - x^2 = r^2 - 2rx$.

In the triangle AHB : $y^2 = (r + x)^2 - (r - x)^2 = 4rx$.

Comparing the two equations: $r^2 = 6rx$, so $x = \frac{r}{6}$.

Part II

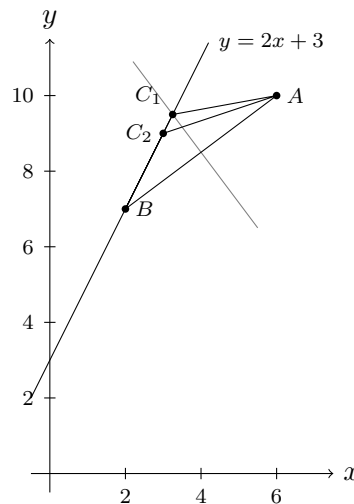
5. Two vertices of a triangle are $A(6, 10)$ and $B(2, 7)$, and the third vertex is on the line $y = 2x + 3$. Find this vertex so that
- the triangle is isosceles with base AB ;
 - the sum of the squares of its sides is minimal.

(16 points)

Solution: a) The third vertex is the intersection of the perpendicular bisector of AB with the given line. The midpoint of AB is $\left(4, \frac{17}{2}\right)$ and $\overrightarrow{AB}$ is parallel to $(4, 3)$, so the perpendicular bisector is

$$4x + 3y = \frac{83}{2}$$

Substituting $y = 2x + 3$: $10x + 9 = \frac{83}{2}$, so the vertex is $C_1\left(\frac{13}{4}, \frac{19}{2}\right)$.



- b) $AB = 5$. Let $C(x, 2x + 3)$. Then

$$AB^2 + AC^2 + BC^2 = 25 + (x - 6)^2 + (2x - 7)^2 + (x - 2)^2 + (2x - 4)^2 = 10x^2 - 60x + 130$$

This quadratic expression has a positive leading coefficient, so it has a minimum, at $x = \frac{60}{2 \cdot 10} = 3$. So the third vertex is $C_2(3, 9)$.

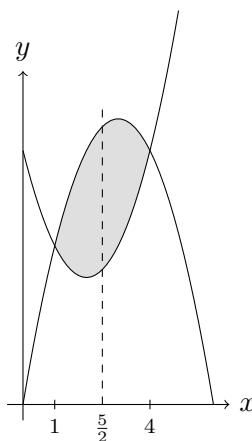
6. a) The line $x = p$ bisects the area of the region bounded by the parabolas $y = -x^2 + 6x$ and $y = x^2 - 4x + 8$. Find the value of p .
- b) We rotate the region bounded by the parabolas $y = -x^2 + 6x$ and $y = x^2 - 4x + 8$ about the x -axis. Find the volume of the resulting solid.

(16 points)

Solution: a) The parabolas intersect where $-x^2 + 6x = x^2 - 4x + 8$, that is, $2x^2 - 10x + 8 = 0$: at $x = 1$ and $x = 4$, in the points $(1, 5)$ and $(4, 8)$. Between them the first parabola is above the second, and the difference is

$$(-x^2 + 6x) - (x^2 - 4x + 8) = -2x^2 + 10x - 8 = -2(x-1)(x-4)$$

which is symmetric about $x = \frac{5}{2}$. So the line $x = \frac{5}{2}$ bisects the area: $p = \frac{5}{2}$. (In fact the region is centrally symmetric about the point $(\frac{5}{2}, \frac{13}{2})$, the midpoint of the vertices $(3, 9)$ and $(2, 4)$ of the two parabolas.)



- b) Both parabolas are above the x -axis on $[1, 4]$, so the volume is

$$\begin{aligned} V &= \pi \int_1^4 (-x^2 + 6x)^2 dx - \pi \int_1^4 (x^2 - 4x + 8)^2 dx = \pi \int_1^4 (-4x^3 + 4x^2 + 64x - 64) dx \\ &= \pi \left[-x^4 + \frac{4}{3}x^3 + 32x^2 - 64x \right]_1^4 = 117\pi \end{aligned}$$

7. In the Csoki Factory, a consumer protection inspection found that a bag contains exactly the prescribed (standard) 40 pieces of candy with probability 0.95, and more or fewer with probability only 0.05.
- a) We choose 5 bags at random. What is the probability that each of them contains exactly 40 pieces of candy?
- b) What is the probability that among the five bags we find at least two that are not standard?
- c) What is the probability that exactly 95 of 100 bags are standard?

(16 points)

Solution: a) Each bag contains 40 pieces with probability 0.95, so the answer is $0.95^5 \approx 0.774$.

b) The complementary event: all bags are standard (probability 0.95^5), or exactly one is not standard (probability $5 \cdot 0.95^4 \cdot 0.05$). So the probability is

$$1 - (0.95^5 + 5 \cdot 0.95^4 \cdot 0.05) \approx 0.0226$$

c) $\binom{100}{95} \cdot 0.95^{95} \cdot 0.05^5 \approx 0.1800$.

8. A company supports a university with 10 million forints. The money is deposited in a bank at 7% annual interest. According to the conditions, the amount must reach 15 million forints, and then in the following 20 years, at the beginning of each year, it must be divided equally among the 10 best first-year students as a scholarship, so that the money runs out with the last payment. The amount earns interest all along, and the annual interest is 7% throughout.
- After how many years do they start paying the scholarships?
 - How much money does each student receive?

(16 points)

Solution: a) Let n be the number of years. We need $10 \cdot 1.07^n \geq 15$, that is,

$$n \geq \frac{\lg 1.5}{\lg 1.07} \approx 5.99$$

So they start paying the scholarships after 6 years.

b) At the beginning of the 7th year the amount is $a = 10\,000\,000 \cdot 1.07^6 \approx 15\,007\,304$ forints. Let x be the total scholarship paid at the beginning of each year. After the first payment $a - x$ remains, which grows to $1.07a - 1.07x$ by the end of the year; after the second payment $1.07a - 1.07x - x$ remains, and so on. After the 20th payment,

$$1.07^{19}a - 1.07^{19}x - 1.07^{18}x - \dots - x = 0, \quad \text{that is,} \quad 1.07^{19}a = x(1.07^{19} + 1.07^{18} + \dots + 1)$$

The sum of the 20 terms in parentheses is $\frac{1.07^{20} - 1}{1.07 - 1} \approx 40.9955$, so $x \approx 1\,323\,910$ forints, and each student receives about 132 391 forints per year.

Note: KoMaL gets 132 393 forints because it rounds the sum to 40.995.

9. The towns A , B , and C , at the same height above sea level, form a triangle: A and B are 53 km apart, B and C are 45 km apart, and A and C are 28 km apart. At the point dividing the segment AB into three equal parts that is closer to A , an 800 m high storm warning tower was built.
- At what angle of elevation is the top of the tower seen from the town A ?
 - How far (in a straight line) is the top of the tower from the town D , which is 20 km from both A and C ?

(16 points)

Solution: Let P be the foot and T the top of the tower; $AP = \frac{53}{3}$ km and $PT = 0.8$ km.

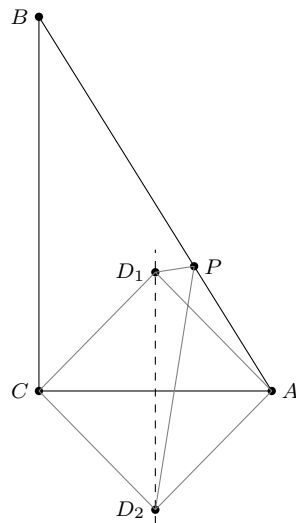
a) In the right triangle TPA :

$$\tan \alpha = \frac{0.8}{\frac{53}{3}} \approx 0.0453, \quad \text{so} \quad \alpha \approx 2.59^\circ$$

b) Since $53^2 = 45^2 + 28^2$, the triangle ABC has a right angle at C . We use coordinates (in km): $C(0, 0)$, $A(28, 0)$, $B(0, 45)$. Then

$$P = A + \frac{1}{3}\overrightarrow{AB} = \left(\frac{56}{3}, 15\right)$$

The town D is on the perpendicular bisector $x = 14$ of AC , with $DA = 20$, so $D(14, \pm\sqrt{204}) \approx (14, \pm 14.28)$. There are two possible positions.



If D is on the same side of AC as B : $PD_1 = \sqrt{\left(\frac{56}{3} - 14\right)^2 + (15 - 14.28)^2} \approx 4.72$ km, and $TD_1 = \sqrt{4.72^2 + 0.8^2} \approx 4.79$ km.

If D is on the other side: $PD_2 = \sqrt{\left(\frac{56}{3} - 14\right)^2 + (15 + 14.28)^2} \approx 29.65$ km, and $TD_2 = \sqrt{29.65^2 + 0.8^2} \approx 29.66$ km.

So the top of the tower is about 4.79 km or about 29.66 km from D .