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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following system of equations over the pairs of integers.

$$\begin{aligned}9^x \cdot 3^y &= 81 \\ 6x + 6y + 5xy &= 0\end{aligned}$$

(11 points)

2. Using the positive integers $1, 2, 3, \dots, n$, we form the following sets: A is the set of the even numbers, B the set of the numbers divisible by 3, and C the set of the numbers divisible by 5.

a) We choose one of the n numbers at random. What is the probability that it was not used in forming the sets, if $n = 48$? (6 points)

b) If $A \cap B \cap C$ has 2 elements, how many elements can $(A \cap B) \setminus C$ have? (7 points)

3. A cattle farm keeps 1000 cows, which give 7 300 000 liters of milk a year. The farm management found that, probably because of the larger pasture per cow, if the number of cows is decreased by 5%, then the annual milk yield per cow increases by 10%.

a) By how much does the annual milk production of the farm increase if the number of cows is decreased by 5%? (5 points)

b) For a herd of this size, the fact that a $p\%$ decrease in the number of cows increases the annual yield per cow by $2p\%$ is valid only for $p < 20$. By what percent should the herd be decreased so that the annual milk production increases by 10%? (8 points)

4. The following is known about the primes $p_1 < p_2 < p_3 < p_4$: $p_3 = p_1 + p_2$, $p_2^2 + 8$ is also a prime, and $p_4 = 2p_1p_2p_3 + p_1 + p_3$. Find the product $p_1p_2p_3p_4$.

(14 points)

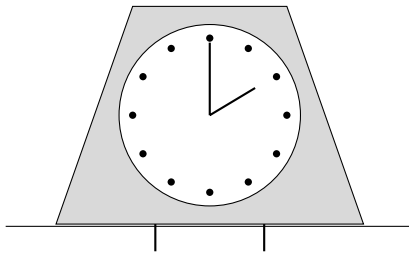
Part II

5. We know about the function $f(x) = x^3 + px^2 + qx$ ($p \neq -2$, $q \neq 0$), defined on the real numbers, that it has exactly two zeros, and $f(2) = -p$.

a) Find the zeros of the function. (8 points)

b) Where does the tangent line of f through the origin intersect the graph of the function again? (8 points)

6. A designer plans a clock mounted on a wooden board in the shape of a symmetric trapezoid. The radius of the circular clock face is 10 cm, its center is 12 cm from every side of the trapezoid, and the longer parallel side is twice the shorter one.



- a) What is the area of the wooden board needed? *(7 points)*
 - b) After 2 o'clock, when is the earliest time that the two hands make the same angle as at 2 o'clock? *(9 points)*
7. In a city zoo, 4 predators (a female tiger, a lion, a leopard, and a jaguar) were placed in 8 cages next to each other in a row.
 - a) In how many ways can the predators be placed in the 8 cages? *(4 points)*
 - b) The keepers found that if two predators are in neighboring cages, they become very aggressive. In how many ways can the predators be placed in the 8 cages so that no two animals are in neighboring cages? *(5 points)*
 - c) The jaguar was taken away, and a male tiger came in its place. So that the female and the male tiger "get closer" to each other, the keepers wanted to place the two tigers in neighboring cages. In how many ways could they place the predators, if the two tigers were in neighboring cages, but otherwise no predators were neighbors? *(7 points)*
 8. The roof of a house with a square base is a right pyramid with a square base. The lateral edges make an angle of 36° with the base square.
 - a) Find the angle between a lateral face of the roof and its base. *(3 points)*
 - b) What is the volume of the attic if the lateral edges are 8 m long? *(4 points)*
 - c) The base square of the roof is surrounded by a rain gutter. The rain falling on the roof flows into the gutter, and from there, through a vertical downspout at one corner of the house, into a barrel in the shape of a right circular cylinder. The barrel has a diameter of 68 cm and a height of 120 cm. During a 15-minute summer storm, according to the news, 20 mm of rain fell per hour. Will the barrel be full of the water that flowed off the roof after the storm? *(9 points)*
 9. a) 12 people signed up for the high school section of a teachers' conference. Two members of the group (the moderators) knew everyone in the group. Five members of the group knew nobody except the moderators, while among the other members of the section everyone knew everyone (acquaintance is always mutual). Before the first session, those who did not know each other introduced themselves with a handshake. How many handshakes took place? *(7 points)*
 - b) The kindergarten section had three moderators; they also knew everyone in the group. Half of the other members of the group knew nobody except the moderators, while among the rest of the group everyone knew everyone (again, acquaintance is mutual). In this section, too, those who did not know each other introduced themselves, and there were 35 handshakes. How many people were in the kindergarten section? *(9 points)*