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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following system of equations over the pairs of integers.

$$\begin{aligned} 9^x \cdot 3^y &= 81 \\ 6x + 6y + 5xy &= 0 \end{aligned}$$

(11 points)

Solution: The first equation is $3^{2x+y} = 3^4$, so $2x + y = 4$, that is, $y = 4 - 2x$. Substituting into the second equation: $6x + 6(4 - 2x) + 5x(4 - 2x) = 0$, that is, $5x^2 - 7x - 12 = 0$. Its solutions are $x_1 = -1$ and $x_2 = \frac{12}{5}$. The second one is not an integer, so the only integer solution is $(x, y) = (-1, 6)$.

2. Using the positive integers $1, 2, 3, \dots, n$, we form the following sets: A is the set of the even numbers, B the set of the numbers divisible by 3, and C the set of the numbers divisible by 5.

a) We choose one of the n numbers at random. What is the probability that it was not used in forming the sets, if $n = 48$? (6 points)

b) If $A \cap B \cap C$ has 2 elements, how many elements can $(A \cap B) \setminus C$ have? (7 points)

Solution: a) The numbers from 1 to 48 divisible by none of 2, 3, 5 are 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47: 13 numbers. So the probability is $\frac{13}{48} \approx 0.271$.

b) The elements of $A \cap B \cap C$ are the numbers divisible by 2, 3, and 5, that is, by 30. If this set has 2 elements, they are 30 and 60, so $60 \leq n < 90$. The elements of $(A \cap B) \setminus C$ are the multiples of 6 not divisible by 5. For $n = 60$ there are 8 of them: 6, 12, 18, 24, 36, 42, 48, 54; as n grows up to 89, we get 4 more: 66, 72, 78, 84. So $8 \leq |(A \cap B) \setminus C| \leq 12$.

3. A cattle farm keeps 1000 cows, which give 7 300 000 liters of milk a year. The farm management found that, probably because of the larger pasture per cow, if the number of cows is decreased by 5%, then the annual milk yield per cow increases by 10%.

a) By how much does the annual milk production of the farm increase if the number of cows is decreased by 5%? (5 points)

b) For a herd of this size, the fact that a $p\%$ decrease in the number of cows increases the annual yield per cow by $2p\%$ is valid only for $p < 20$. By what percent should the herd be decreased so that the annual milk production increases by 10%? (8 points)

Solution: a) The annual yield of one cow is $\frac{7\,300\,000}{1000} = 7300$ liters. After the decrease, the annual production is $1000 \cdot 0.95 \cdot 7300 \cdot 1.1 = 7\,628\,500$ liters, an increase of $7\,628\,500 - 7\,300\,000 = 328\,500$ liters.

b) After a $p\%$ decrease there are $1000 \left(1 - \frac{p}{100}\right)$ cows, each giving $7300 \left(1 + \frac{2p}{100}\right)$ liters. For a 10% increase,

$$1000 \left(1 - \frac{p}{100}\right) \cdot 7300 \left(1 + \frac{2p}{100}\right) = 7\,300\,000 \cdot 1.1$$

that is, $1 + \frac{p}{100} - \frac{2p^2}{100^2} = 1.1$, so $p^2 - 50p + 500 = 0$. The root less than 20 is $p = 25 - 5\sqrt{5} \approx 13.8$. So the herd should be decreased by about 13.8%.

4. The following is known about the primes $p_1 < p_2 < p_3 < p_4$: $p_3 = p_1 + p_2$, $p_2^2 + 8$ is also a prime, and $p_4 = 2p_1p_2p_3 + p_1 + p_3$. Find the product $p_1p_2p_3p_4$.

(14 points)

Solution: Every prime other than 2 is odd, so from $p_3 = p_1 + p_2$ one of the primes on the right side must be even; since p_1 is the smallest, $p_1 = 2$.

If $p_2 = 3k+1$, then $p_2^2+8 = 9k^2+6k+9$ is divisible by 3, so it is not a prime. If $p_2 = 3k+2$, then $p_2^2+8 = 9k^2+12k+12$ is divisible by 3, so it is not a prime either. So p_2 is divisible by 3, that is, $p_2 = 3$ (and indeed $3^2 + 8 = 17$ is a prime).

Then $p_3 = 2 + 3 = 5$, and $p_4 = 2 \cdot 2 \cdot 3 \cdot 5 + 2 + 5 = 67$. The product is $p_1p_2p_3p_4 = 2 \cdot 3 \cdot 5 \cdot 67 = 2010$.

Part II

5. We know about the function $f(x) = x^3 + px^2 + qx$ ($p \neq -2$, $q \neq 0$), defined on the real numbers, that it has exactly two zeros, and $f(2) = -p$.

a) Find the zeros of the function. (8 points)

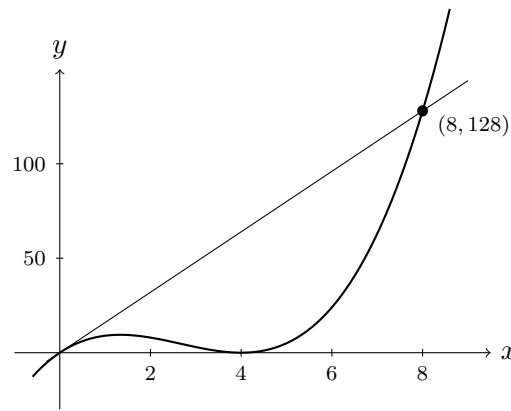
b) Where does the tangent line of f through the origin intersect the graph of the function again? (8 points)

Solution: a) $x^3 + px^2 + qx = x(x^2 + px + q) = 0$, so one zero is 0. Since $q \neq 0$, 0 is not a zero of $x^2 + px + q$, so this quadratic must have exactly one (double) zero: $p^2 - 4q = 0$. From $f(2) = -p$: $8 + 4p + 2q = -p$, that is, $5p + 2q = -8$. With $2q = \frac{p^2}{2}$: $p^2 + 10p + 16 = 0$, so $p = -8$ or $p = -2$. Since $p \neq -2$, $p = -8$ and $q = 16$:

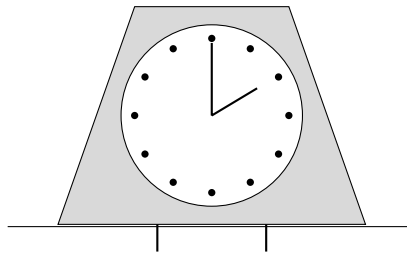
$$f(x) = x^3 - 8x^2 + 16x = x(x-4)^2$$

The zeros are $x_1 = 0$ and $x_2 = 4$.

b) $f'(x) = 3x^2 - 16x + 16$, and $f'(0) = 16$, so the tangent line at the origin is $y = 16x$. It meets the graph where $x^3 - 8x^2 + 16x = 16x$, that is, $x^2(x-8) = 0$. So the tangent line intersects the graph again at $x = 8$, at the point $(8, 128)$.

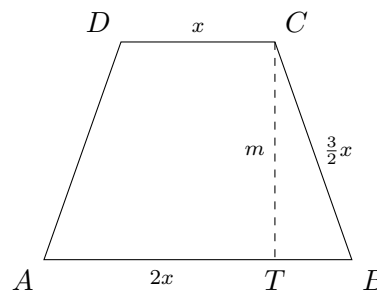


6. A designer plans a clock mounted on a wooden board in the shape of a symmetric trapezoid. The radius of the circular clock face is 10 cm, its center is 12 cm from every side of the trapezoid, and the longer parallel side is twice the shorter one.



- a) What is the area of the wooden board needed? (7 points)
 b) After 2 o'clock, when is the earliest time that the two hands make the same angle as at 2 o'clock? (9 points)

Solution: a) Enlarging the clock face from its center by the ratio $\frac{12}{10} = \frac{6}{5}$, we get a circle that touches every side of the trapezoid. So the trapezoid is a tangential quadrilateral. If the shorter parallel side is x , then the longer one is $2x$, and by the theorem on tangential quadrilaterals the legs are $\frac{x + 2x}{2} = \frac{3}{2}x$.



In the right triangle BTC : $BC = \frac{3}{2}x$, $BT = \frac{2x-x}{2} = \frac{1}{2}x$, and $CT = m = 2 \cdot 12 = 24$. By the Pythagorean theorem,

$$\left(\frac{3}{2}x\right)^2 = \left(\frac{1}{2}x\right)^2 + 24^2, \quad \text{that is,} \quad 2x^2 = 576$$

so $x = 12\sqrt{2}$. The area of the trapezoid is

$$T = \frac{(24\sqrt{2} + 12\sqrt{2}) \cdot 24}{2} = 432\sqrt{2} \approx 610.9 \text{ cm}^2$$

b) At 2 o'clock the two hands make an angle of 60° . If the hour hand turns by α until the next time the hands make 60° (now with the minute hand ahead), then the minute hand turns by $60^\circ + \alpha + 60^\circ = 120^\circ + \alpha$. The minute hand is 12 times as fast as the hour hand, so $12\alpha = 120^\circ + \alpha$, that is, $\alpha = \frac{120^\circ}{11}$. The hour hand turns 30° in 60 minutes, that is, 1° in 2 minutes, so this takes

$$2 \cdot \frac{120}{11} = \frac{240}{11} \approx 21.82 \text{ minutes}$$

So the earliest time after 2 o'clock is about 2:21:49.

Note: KoMaL gives 2 hours 21 minutes 48 seconds; since $\frac{240}{11}$ minutes is 21 minutes 49.1 seconds, the time is closer to 2:21:49.

7. In a city zoo, 4 predators (a female tiger, a lion, a leopard, and a jaguar) were placed in 8 cages next to each other in a row.

a) In how many ways can the predators be placed in the 8 cages? (4 points)

b) The keepers found that if two predators are in neighboring cages, they become very aggressive. In how many ways can the predators be placed in the 8 cages so that no two animals are in neighboring cages? (5 points)

c) The jaguar was taken away, and a male tiger came in its place. So that the female and the male tiger "get closer" to each other, the keepers wanted to place the two tigers in neighboring cages. In how many ways could they place the predators, if the two tigers were in neighboring cages, but otherwise no predators were neighbors? (7 points)

Solution: a) We can choose the 4 cages in $\binom{8}{4}$ ways and place the 4 animals into them in $4!$ ways:

$$\binom{8}{4} \cdot 4! = \frac{8!}{4!} = 5 \cdot 6 \cdot 7 \cdot 8 = 1680$$

b) We count the choices of 4 cages with no two neighbors. If cage 1 is used, the other three can be cages 3, 5, 7; or 3, 5, 8; or 3, 6, 8; or 4, 6, 8. If cage 1 is not used, only 2, 4, 6, 8 is possible. That is 5 choices, and in each the animals can be placed in $4!$ ways: $5 \cdot 4! = 120$ ways.

c) The two tigers can be in cages 1–2, 2–3, 3–4, 4–5, 5–6, 6–7, or 7–8. For each position we count the choices of two more cages, not neighboring each other or the tigers:

tigers in 1–2: 6 choices (from cages 4–8); tigers in 2–3: 3 choices (from cages 5–8); tigers in 3–4: 4 choices; tigers in 4–5: 4 choices; and by symmetry 5–6 is like 3–4, 6–7 like 2–3, and 7–8 like 1–2.

In each case the two tigers can switch places, and so can the other two animals, so each choice gives $2 \cdot 2 = 4$ arrangements. In all,

$$4(6 + 3 + 4 + 4 + 4 + 3 + 6) = 24 + 12 + 16 + 16 + 16 + 12 + 24 = 120$$

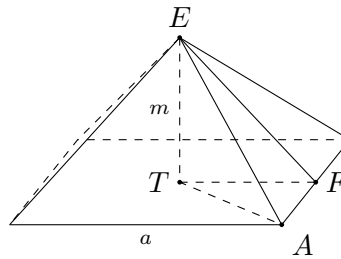
8. The roof of a house with a square base is a right pyramid with a square base. The lateral edges make an angle of 36° with the base square.

a) Find the angle between a lateral face of the roof and its base. (3 points)

b) What is the volume of the attic if the lateral edges are 8 m long? (4 points)

c) The base square of the roof is surrounded by a rain gutter. The rain falling on the roof flows into the gutter, and from there, through a vertical downspout at one corner of the house, into a barrel in the shape of a right circular cylinder. The barrel has a diameter of 68 cm and a height of 120 cm. During a 15-minute summer storm, according to the news, 20 mm of rain fell per hour. Will the barrel be full of the water that flowed off the roof after the storm? (9 points)

Solution: a) Let the side of the base square be a , the height m , T the center of the base, A a vertex, F the midpoint of a side, and α the angle between a lateral face and the base.



In the right triangle ATE , $AT = \frac{a\sqrt{2}}{2}$, so $\tan 36^\circ = \frac{m}{\frac{a\sqrt{2}}{2}} = \frac{m\sqrt{2}}{a}$. In the right triangle EFT , $\tan \alpha = \frac{m}{\frac{a}{2}} = \frac{2m}{a}$.

Dividing, $\frac{\tan \alpha}{\tan 36^\circ} = \frac{2}{\sqrt{2}} = \sqrt{2}$, so $\tan \alpha = \sqrt{2} \tan 36^\circ$, and $\alpha \approx 45.78^\circ$.

b) If the lateral edge is 8, then from the triangle ATE : $AT = 8 \cos 36^\circ \approx 6.47$ m, so the side of the base is $a = AT\sqrt{2} \approx 9.15$ m, and the height is $m = 8 \sin 36^\circ \approx 4.70$ m. The volume of the attic is

$$V = \frac{a^2 m}{3} \approx 131.3 \text{ m}^3$$

(KoMaL, rounding a and m first, gets 131.17 m^3 .)

c) The radius of the barrel is 0.34 m and its height 1.2 m, so its volume is $V_h = 0.34^2 \pi \cdot 1.2 \approx 0.4358 \text{ m}^3$. 20 mm of rain per hour means 5 mm in 15 minutes. The rain falling on the roof is the rain falling on a 9.15 m square, so its volume is that of a prism with this square base and height 0.005 m: $9.15^2 \cdot 0.005 \approx 0.4186 \text{ m}^3$. This is less than 0.4358 m^3 , so the barrel will not be full.

9. a) 12 people signed up for the high school section of a teachers' conference. Two members of the group (the moderators) knew everyone in the group. Five members of the group knew nobody except the moderators, while among the other members of the section everyone knew everyone (acquaintance is always mutual). Before the first session, those who did not know each other introduced themselves with a handshake. How many handshakes took place? (7 points)
- b) The kindergarten section had three moderators; they also knew everyone in the group. Half of the other members of the group knew nobody except the moderators, while among the rest of the group everyone knew everyone (again, acquaintance is mutual). In this section, too, those who did not know each other introduced themselves, and there were 35 handshakes. How many people were in the kindergarten section? (9 points)

Solution: a) We draw the acquaintance graph of the 12 people: A and B are the moderators, C, D, E, F, G know nobody else, and H, I, J, K, L are the others. We need the number of edges missing from the complete graph. The degrees are 11 for A and B , 2 for each of C, D, E, F, G , and 6 for each of H, I, J, K, L . The sum of the degrees is $2 \cdot 11 + 5 \cdot 2 + 5 \cdot 6 = 62$, so the graph has 31 edges. The complete graph on 12 vertices has $\frac{12 \cdot 11}{2} = 66$ edges, so 35 edges are missing: there were 35 handshakes.

b) Let k be the number of people who knew nobody except the moderators; then k people in the group knew everyone, and the graph has $2k + 3$ vertices. The three moderators and the k people who knew everyone form a complete graph on $k + 3$ vertices, with $\frac{(k+3)(k+2)}{2}$ edges; the people who knew only the moderators give $3k$ more edges. The complete graph on $2k + 3$ vertices has $\frac{(2k+3)(2k+2)}{2}$ edges, so the number of missing edges is

$$\frac{(2k+3)(2k+2)}{2} - \frac{(k+3)(k+2)}{2} - 3k = 35$$

which simplifies to $3k^2 - k - 70 = 0$. The negative solution is not meaningful, so $k = 5$, and the kindergarten section had $2k + 3 = 13$ members.