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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation over the integers.

$$|45x - 8| + 7 \cdot 2^{x+4} + \log_2^2(x - 2) + \sqrt{4x + 2009} = 2010$$

(11 points)

2. One side of a triangle is $a = 10$, and the two angles on it are $\beta = 50^\circ$ and $\gamma = 60^\circ$. Find the volume of the solid obtained by rotating the triangle about its longest side.

(12 points)

3. a) Find the elements of the following sets.

A : the integer roots of the inequality $\log_3(x - 6) < 2$;

B : the positive integers less than 20 that have at least 4 divisors;

C : the possible values of the digit sum of three-digit numbers whose digits form an arithmetic sequence.

b) Find the elements of the set $(C \setminus A) \cup (A \cap B)$.

(14 points)

4. For which elements x of the interval $[-2, 2]$ are $\sin x$, $2 \tan 2x$, and $3 \cos x$ consecutive terms of a geometric sequence, in this order?

(14 points)

Part II

5. The point A lies on the line e : $2x - y + 4 = 0$, and the point B on the line f : $2x + 3y - 8 = 0$. Find the coordinates of A and B if the midpoint of the segment AB is $F(5, 4)$.

(16 points)

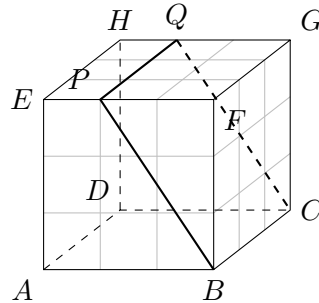
6. a) Find the coordinates of the points P and Q , if P is the inflection point of the function $f(x) = x^3 - 3x^2 - 9x + 2$, the first coordinate of Q is the local maximum point of f , and its second coordinate is the local maximum value.

b) Find the formula of the quadratic function g whose graph has its vertex at P and passes through Q .

c) Find the area of the region enclosed by the graphs of the two functions with vertices P and Q .

(16 points)

7. We paint 3 faces of some congruent cubes of edge 1 red, all cubes the same way, so that the three painted faces meet at a common vertex. From 8 such cubes we build a $2 \times 2 \times 2$ larger cube, placing the small cubes on each other at random.
- What is the probability that every face of the large cube is (completely) red?
 - What would this probability be if we built a $3 \times 3 \times 3$ large cube from 27 small cubes?
 - We cut the assembled $3 \times 3 \times 3$ cube $ABCDEFGH$ by the plane $BCQP$ (P and Q are the points of the edges EF and HG dividing them into three equal parts, as in the figure), and then we take the whole structure apart. We choose one of the resulting solids at random, with equal probability. What is the probability that we choose a small cube?



(16 points)

8. Which point of the graph of the function $f : (-14, 5] \rightarrow \mathbb{R}$, $f(x) = |x| + |x + 2| - |x - 3|$ is closest to, and which is farthest from, the point $P(-4, 7)$?

(16 points)

9. Two consecutive natural numbers, Larger (**L**) and Smaller (**S**), are talking:

S: I have 6 divisors.

L: I have more.

S: The sum of my digits is 11.

L: Mine is less.

S: Exactly two of my digits are the same.

L: Mine too!

Which are these two numbers?

(16 points)