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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation over the integers.

$$|45x - 8| + 7 \cdot 2^{x+4} + \log_2^2(x - 2) + \sqrt{4x + 2009} = 2010$$

(11 points)

Solution: Because of the logarithm, $x > 2$; since x is an integer, $x \geq 3$. Every term on the left side is nonnegative, so $x \leq 4$: for $x = 5$ the term $7 \cdot 2^{x+4}$ is already more than 2010, and it is even larger for larger x . So only $x = 3$ and $x = 4$ are possible, and substituting, the only integer solution is $x = 4$ ($172 + 1792 + 1 + 45 = 2010$).

2. One side of a triangle is $a = 10$, and the two angles on it are $\beta = 50^\circ$ and $\gamma = 60^\circ$. Find the volume of the solid obtained by rotating the triangle about its longest side.

(12 points)

Solution: From the given angles $\alpha = 70^\circ$, so the longest side is a , and we rotate the triangle about it. The solid consists of two cones with a common base circle, whose radius is the altitude m_a . Computing the area of the triangle in two ways,

$$\frac{a \cdot m_a}{2} = \frac{a^2 \sin \beta \sin \gamma}{2 \sin \alpha}, \quad \text{so} \quad m_a \approx 7.06$$

If the heights of the cones are m_1 and m_2 , then $m_1 + m_2 = 10$, and the volume is

$$V = \frac{m_a^2 \pi \cdot m_1}{3} + \frac{m_a^2 \pi \cdot m_2}{3} = \frac{m_a^2 \pi}{3} \cdot 10 \approx 521.95$$

3. a) Find the elements of the following sets.

A : the integer roots of the inequality $\log_3(x - 6) < 2$;

B : the positive integers less than 20 that have at least 4 divisors;

C : the possible values of the digit sum of three-digit numbers whose digits form an arithmetic sequence.

b) Find the elements of the set $(C \setminus A) \cup (A \cap B)$.

(14 points)

Solution: a) For the set A , by the definition of the logarithm, $0 < x - 6 < 3^2$, so $A = \{7, 8, 9, 10, 11, 12, 13, 14\}$. By the description, $B = \{6, 8, 10, 12, 14, 15, 16, 18\}$. For the set C : the digit sum is 3 times the middle digit, so the elements are divisible by 3; the smallest is 3 (in 111) and the largest 27 (in 999), and every multiple of 3 in between occurs: $C = \{3, 6, 9, 12, 15, 18, 21, 24, 27\}$.

b) $C \setminus A = \{3, 6, 15, 18, 21, 24, 27\}$ and $A \cap B = \{8, 10, 12, 14\}$, so

$$(C \setminus A) \cup (A \cap B) = \{3, 6, 8, 10, 12, 14, 15, 18, 21, 24, 27\}$$

4. For which elements x of the interval $[-2, 2]$ are $\sin x$, $2 \tan 2x$, and $3 \cos x$ consecutive terms of a geometric sequence, in this order?

(14 points)

Solution: The terms of a geometric sequence cannot be 0, so $\sin x \neq 0$, $\cos x \neq 0$, and $\tan 2x \neq 0$, that is, $x \neq k\frac{\pi}{2}$, $k \in \mathbb{Z}$. The square of the middle term equals the product of its neighbors, so $4 \tan^2 2x = 3 \sin x \cos x$, that is,

$$8 \tan^2 2x = 3 \cdot 2 \sin x \cos x, \quad 8 \frac{\sin^2 2x}{\cos^2 2x} = 3 \sin 2x$$

By the condition we can divide by $\sin 2x$; after rearranging (using $\cos^2 2x = 1 - \sin^2 2x$), $3 \sin^2 2x + 8 \sin 2x - 3 = 0$, so $\sin 2x = -3$ (impossible) or $\sin 2x = \frac{1}{3}$.

$2x \approx 0.340 + 2k_1\pi$, that is, $x \approx 0.170 + k_1\pi$; only $k_1 = 0$ gives a root in $[-2, 2]$.

$2x \approx 2.802 + 2k_2\pi$, that is, $x \approx 1.401 + k_2\pi$; $k_2 = 0$ and $k_2 = -1$ give roots in $[-2, 2]$.

So the three solutions are $x_1 \approx 0.170$, $x_2 \approx 1.401$, and $x_3 \approx -1.741$; they can be checked to satisfy the conditions.

Part II

5. The point A lies on the line $e: 2x - y + 4 = 0$, and the point B on the line $f: 2x + 3y - 8 = 0$. Find the coordinates of A and B if the midpoint of the segment AB is $F(5, 4)$.

(16 points)

Solution: Reflecting the line e in the point F , the point A goes to B . So if we reflect e in F , the image meets f at B . Two points of e are $(0, 4)$ and $(1, 6)$; their images in F are $(10, 4)$ and $(9, 2)$, so the image line is $e': 2x - y = 16$. Solving the system of e' and f , we get $B(7, -2)$, and the reflection of B in F gives $A(3, 10)$.

6. a) Find the coordinates of the points P and Q , if P is the inflection point of the function $f(x) = x^3 - 3x^2 - 9x + 2$, the first coordinate of Q is the local maximum point of f , and its second coordinate is the local maximum value.
b) Find the formula of the quadratic function g whose graph has its vertex at P and passes through Q .
c) Find the area of the region enclosed by the graphs of the two functions with vertices P and Q .

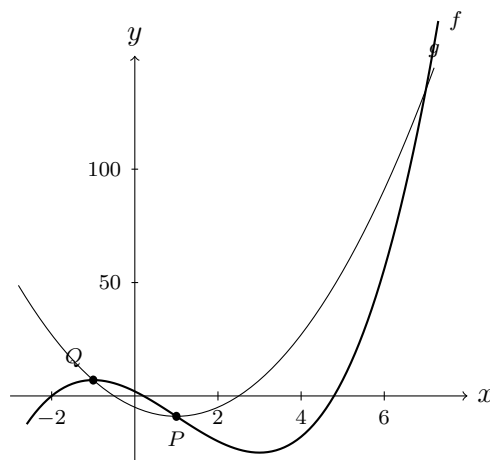
(16 points)

Solution: a) $f'(x) = 3x^2 - 6x - 9$, whose zeros are -1 and 3 . Since the leading coefficient is positive, f increases until its first extremum, so -1 is the local maximum point, and $f(-1) = 7$. $f''(x) = 6x - 6$, whose zero is 1 , and $f(1) = -9$. So $P(1, -9)$ and $Q(-1, 7)$.

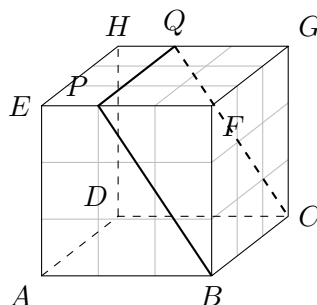
b) The vertex of the parabola is $P(1, -9)$, so $g(x) = a(x - 1)^2 - 9$. Substituting Q : $7 = a \cdot (-2)^2 - 9$, so $a = 4$, and $g(x) = 4(x - 1)^2 - 9 = 4x^2 - 8x - 5$.

c) $f(x) - g(x) = x^3 - 7x^2 - x + 7 = (x - 7)(x - 1)(x + 1)$, so the graphs meet at $x = -1, 1, 7$, and the required area is

$$\left| \int_{-1}^1 (x^3 - 7x^2 - x + 7) dx \right| = \left| \left[\frac{x^4}{4} - \frac{7x^3}{3} - \frac{x^2}{2} + 7x \right]_{-1}^1 \right| = \frac{28}{3}$$



7. We paint 3 faces of some congruent cubes of edge 1 red, all cubes the same way, so that the three painted faces meet at a common vertex. From 8 such cubes we build a $2 \times 2 \times 2$ larger cube, placing the small cubes on each other at random.
- What is the probability that every face of the large cube is (completely) red?
 - What would this probability be if we built a $3 \times 3 \times 3$ large cube from 27 small cubes?
 - We cut the assembled $3 \times 3 \times 3$ cube $ABCDEFGH$ by the plane $BCQP$ (P and Q are the points of the edges EF and HG dividing them into three equal parts, as in the figure), and then we take the whole structure apart. We choose one of the resulting solids at random, with equal probability. What is the probability that we choose a small cube?



(16 points)

Solution: a) A cube can be placed in 24 ways (any face can be at the bottom, and then it can be turned in 4 ways). For every face of the $2 \times 2 \times 2$ cube to be red, each small cube must stand so that its 3 painted faces are outside. This happens in 3 of the 24 cases (the vertex where the 3 red faces meet is determined, and then the cube can be turned in 3 ways). So the probability that a small cube stands in a good position is $\frac{3}{24} = \frac{1}{8}$, and for the whole cube

$$\left(\frac{1}{8}\right)^8 \approx 5.96 \cdot 10^{-8}$$

b) As in a), the small cubes at the 8 corners of the large cube stand in a good position with probability $\frac{1}{8}$. The 6 small cubes at the centers of the faces stand in a good position with probability $\frac{1}{2}$, and the 12 small cubes at the middles of the edges with probability $\frac{1}{4}$ (6 of the 24 cases). The cube in the center can be in any position. So the probability is

$$\left(\frac{1}{8}\right)^8 \cdot \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{4}\right)^{12} = \frac{1}{2^{54}} \approx 5.55 \cdot 10^{-17}$$

c) The plane $BCQP$ cuts 12 small cubes in two (3 on the top level, 6 in the middle, and 3 at the bottom). So after taking the structure apart, $27 - 12 = 15$ small cubes remain, together with $2 \cdot 12 = 24$ solids that are not cubes. The probability of choosing a small cube is

$$\frac{15}{39} = \frac{5}{13} \approx 0.385$$

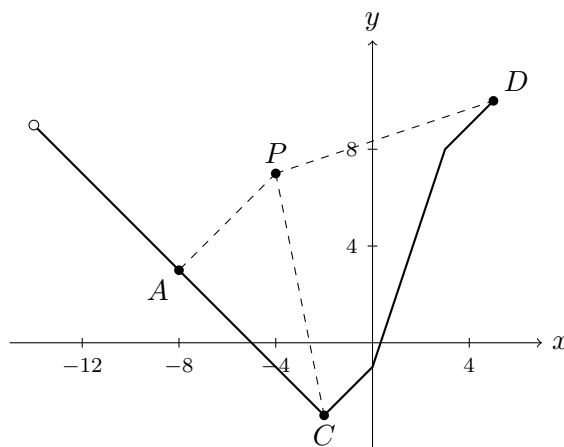
Note: KoMaL writes $25 - 12 = 13$ remaining cubes and gets $\frac{13}{37} \approx 0.351$; but the large cube consists of 27 small cubes.

8. Which point of the graph of the function $f : (-14, 5] \rightarrow \mathbb{R}$, $f(x) = |x| + |x + 2| - |x - 3|$ is closest to, and which is farthest from, the point $P(-4, 7)$?

(16 points)

Solution: It is useful to graph the function; we break it into linear pieces:

$$f(x) = \begin{cases} -x - 5, & \text{if } -14 < x < -2 \\ x - 1, & \text{if } -2 \leq x < 0 \\ 3x - 1, & \text{if } 0 \leq x < 3 \\ x + 5, & \text{if } 3 \leq x \leq 5 \end{cases}$$



The point closest to P is found by dropping perpendiculars to the pieces of the graph. The foot on the first piece is $A(-8, 3)$, with $PA = \sqrt{32}$; every point of the other pieces is farther (the closest one there is at distance $\sqrt{40}$). So the point closest to P is $A(-8, 3)$.

For the farthest point we compute the distances of the two endpoints of the broken line and of the corner C : $C(-2, -3)$ and $(-14, 9)$ are both at distance $\sqrt{104}$ from P , but $(-14, 9)$ does not belong to the graph. The endpoint $D(5, 10)$ is at distance $\sqrt{90}$. So the farthest point is $C(-2, -3)$.

9. Two consecutive natural numbers, Larger (L) and Smaller (S), are talking:

S: I have 6 divisors.

L: I have more.

S: The sum of my digits is 11.

L: Mine is less.

S: Exactly two of my digits are the same.

L: Mine too!

Which are these two numbers?

(16 points)

Solution: The larger number has a smaller digit sum only if the smaller number ends in 9 (otherwise the digit sum of the larger number would be 1 more). Then the digit sum of the larger number can only be $11 - 9 + 1 = 3$, since a 0 takes the place of the 9 in the units, and the tens digit increases by 1. (It could decrease more if the smaller number ended in more 9's, but then its digit sum would be more than 11.)

So the larger number ends in 0 and has digit sum 3. Since exactly two of its digits are the same, it has one more 0, one 1, and one 2. It cannot end in two 0's, so the only possibilities are 1020 and 2010. The smaller number would be 1019 or 2009. $2009 = 7^2 \cdot 41$ has indeed 6 divisors (and 2010 has more), but 1019 is a prime, so it does not have 6 divisors. So the two numbers are 2009 and 2010.