

author: István Darnai

translated from: KoMaL, February 2009, pages 82-83

Pdf file: KoMaL, February 2009, pages 82-83

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. A soccer team has a squad of 20 players: 3 goalkeepers, 6 defenders, 7 midfielders, and 4 forwards. In one game 1 goalkeeper, 4 defenders, 3 midfielders, and 3 forwards will play.

a) In how many ways can the coach choose the starting team? (6 points)

b) The coach wants to get at least 40 points in the first 15 games. In how many ways can this happen if the order of the points earned does not matter? (A win is worth 3 points, a draw 1 point, a loss 0 points.) (5 points)

Solution: a) We can choose 1 of 3 goalkeepers in $\binom{3}{1} = 3$ ways, 4 of 6 defenders in $\binom{6}{4} = 15$ ways, 3 of 7 midfielders in $\binom{7}{3} = 35$ ways, and 3 of 4 forwards in $\binom{4}{3} = 4$ ways. In all, $3 \cdot 15 \cdot 35 \cdot 4 = 6300$ ways.

b) The possible results (wins, draws, losses) and the total points:

wins	draws	losses	points
15	0	0	45
14	1	0	43
14	0	1	42
13	2	0	41
13	1	1	40

13 wins and 2 losses give 39 points, and with fewer than 13 wins at most 39 points can be earned (12 wins, 3 draws). So there are 5 ways to get at least 40 points.

2. The half-year math grades of a class were the following:

grade	1	2	3	4	5
frequency	3	10	7	8	

(In Hungary, 5 is the best grade and 1 is failing.)

a) How many students got a 5, if we know that the class average is between 3.2 and 3.3, and the number of students in the class is even? (9 points)

b) Find the mode and the median of the grades. (4 points)

Solution: a) Let x be the number of 5's. The average is

$$\bar{x} = \frac{3 \cdot 1 + 10 \cdot 2 + 7 \cdot 3 + 8 \cdot 4 + 5x}{3 + 10 + 7 + 8 + x} = \frac{76 + 5x}{28 + x}$$

From $3.2 < \frac{76 + 5x}{28 + x}$ we get $7.56 < x$, and from $\frac{76 + 5x}{28 + x} < 3.3$ we get $x < 9.65$. So $x = 8$ or $x = 9$, and since the class size $28 + x$ is even, $x = 8$: 8 students got a 5, and the class has 36 students. (The average is $\frac{116}{36} \approx 3.22$.)

b) The mode is the most frequent grade, 2. The median is the average of the 18th and 19th grades in increasing order, 3.

3. At the five-year reunion of a class of 36 students, 32 showed up. It turned out that the three most popular summer destinations had been Transylvania, Zemplén, and the Őrség. 18 had spent a summer in Transylvania, 21 in the Őrség, and 17 chose Zemplén. 11 had been in both Transylvania and the Őrség, 9 in both the Őrség and Zemplén, and 8 in both Transylvania and Zemplén. The number of those who had been to all three places equals the number of those who had been to none of them.

a) Draw a Venn diagram.

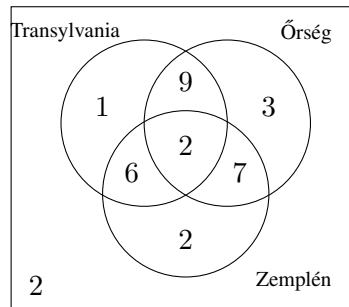
(9 points)

b) What is the probability that a randomly chosen person (of the 32) has been to at least two of these places? (4 points)

Solution: a) Let x be the number who had been to all three places (and also the number who had been to none). By the inclusion–exclusion principle,

$$18 + 21 + 17 - 11 - 8 - 9 + x + x = 32$$

so $x = 2$, and the Venn diagram can be drawn.



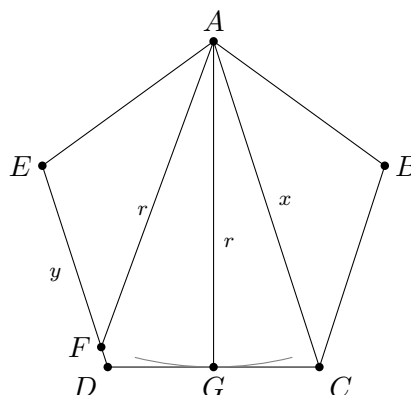
b) The number who have been to at least two places is $9 + 6 + 7 + 2 = 24$, so the probability is

$$P = \frac{24}{32} = \frac{3}{4}$$

4. $ABCDE$ is a regular pentagon with sides 10 cm, and k is the circle with center A that is tangent to the side CD . Let F be the intersection point of the circle and DE . Find the distance EF .

(14 points)

Solution: Let $a = 10$ cm be the side, $x = AC$ the diagonal, and r the radius; G is the midpoint of CD , the point of tangency.



The angles of a regular pentagon are 108° . By the Law of Cosines in the triangle ABC , $x^2 = 100 + 100 - 200 \cos 108^\circ$, so $x \approx 16.18$. In the right triangle AGC , $r^2 = x^2 - (\frac{a}{2})^2$, so $r \approx 15.39$. In the triangle AEF , $AE = a$, $AF = r$, and the angle at E is 108° . By the Law of Sines,

$$\frac{\sin \varphi}{a} = \frac{\sin 108^\circ}{r}, \quad \text{so} \quad \sin \varphi = \frac{10 \sin 108^\circ}{15.39} \approx 0.6180, \quad \varphi \approx 38.17^\circ$$

where φ is the angle at F . The angle at A is $\delta = 180^\circ - 108^\circ - 38.17^\circ = 33.83^\circ$. By the Law of Sines again,

$$\frac{y}{\sin \delta} = \frac{a}{\sin \varphi}, \quad \text{so} \quad y = \frac{10 \sin 33.83^\circ}{\sin 38.17^\circ} \approx 9.01$$

The distance EF is about 9.01 cm.

Part II

5. We want to make a mug of clay in the shape of a truncated cone. The inner diameter of the bottom is 6 cm, that of the top is 9 cm, and the side of the mug makes an angle of 80° with the horizontal.

a) How many deciliters of liquid fit into the mug?

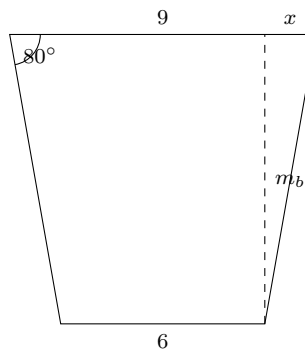
(6 points)

The outer diameter of the top is 9.3 cm, the wall of the mug is equally thick everywhere, the bottom is 4 mm thick, and the handle is made of a cylinder of diameter 1.5 cm and height 15 cm. The density of the clay is 2.3 g/cm^3 .

b) What is the total mass of 10 000 mugs? (Give the mass in kg.)

(10 points)

Solution: a) The inner radii of the mug are $R_b = 4.5$ and $r_b = 3$, so $x = R_b - r_b = 1.5$.



From $\tan 80^\circ = \frac{m_b}{1.5}$, the inner height is $m_b = 1.5 \tan 80^\circ \approx 8.51$, and the volume of the mug is

$$V_b = \frac{(R_b^2 + R_b r_b + r_b^2) m_b \pi}{3} = \frac{(4.5^2 + 4.5 \cdot 3 + 3^2) \cdot 8.51 \pi}{3} \approx 381 \text{ cm}^3$$

So about 3.81 dl fits into the mug.

b) The outer radius of the top is $R_k = 4.65$, and the outer height is $m_k = m_b + 0.4 = 8.91$. From $\tan 80^\circ = \frac{8.91}{y}$, $y \approx 1.57$, so the outer radius of the bottom is $r_k = 4.65 - 1.57 = 3.08$, and

$$V_k = \frac{(4.65^2 + 4.65 \cdot 3.08 + 3.08^2) \cdot 8.91 \pi}{3} \approx 423.9 \text{ cm}^3$$

The volume of the handle is $0.75^2 \pi \cdot 15 \approx 26.5 \text{ cm}^3$. The volume of clay in one mug is $423.9 - 381 + 26.5 = 69.4 \text{ cm}^3$, in 10 000 mugs $694\,000 \text{ cm}^3$, and their mass is $694\,000 \cdot 2.3 = 1\,596\,200 \text{ g}$, about 1596 kg.

6. a) Find the value of p for which the expression $9x^2 + (p + 16)x + 2.4p + 1$ is a perfect square. (9 points)

b) Evaluate

$$\int_3^5 (9x^2 + 66x + 121) dx$$

(7 points)

Solution: a) The expression is a perfect square exactly when its two zeros are equal, that is, when the discriminant is 0:

$$D = (p + 16)^2 - 4 \cdot 9 \cdot (2.4p + 1) = 0, \quad \text{that is,} \quad p^2 - 54.4p + 220 = 0$$

$$p_{1,2} = \frac{54.4 \pm \sqrt{2959.36 - 880}}{2} = \frac{54.4 \pm 45.6}{2}$$

The expression is a perfect square for $p_1 = 50$ and $p_2 = 4.4$. (For $p = 50$ it is $(3x + 11)^2$.)

b)

$$\int_3^5 (9x^2 + 66x + 121) dx = [3x^3 + 33x^2 + 121x]_3^5 = 1805 - 741 = 1064$$

7. In a country of 15 million people, a company studies the ratings of television programs. In 1500 households the TV sets were equipped with a device that always records who watched which channel and when. From the 3200 people observed in this way, whose distribution by sex and age reflects the population, the ratings can be computed. The table shows the distribution of the 3200 people in the sample by age and sex.

	0–17	18–27	28–38	39–49	50–65	66–
women	540		240	220	290	110
men	520		230	210	270	

- a) Fill in the missing data of the table, if we know that the number of men older than 65 is 375 000, and there are 100 more women than men among the people studied. (6 points)

- b) 70 members of the sample watched the Saturday morning children's program. How many viewers does this mean nationwide? (2 points)

- c) If 1.7 million people were watching TV at that time, what percent of the viewers watched the children's program? (2 points)

- d) A program's rating is good if at least 25% of the people watching TV at that time watch it. 504 members of the sample watched a Friday evening quiz show. Is the rating of the show good, if 60% of the population was watching TV at that time? (6 points)

Solution: a) $\frac{375\,000}{15\,000\,000} \cdot 3200 = 80$, so there are 80 men older than 65 in the sample. From $x + (x + 100) = 3200$, there are 1550 men and 1650 women in the sample. So

$$1550 - 520 - 230 - 210 - 270 - 80 = 240, \quad 1650 - 540 - 240 - 220 - 290 - 110 = 250$$

and the missing data are: women aged 18–27: 250; men aged 18–27: 240; men older than 65: 80.

- b) $\frac{70}{3200} \cdot 15\,000\,000 = 328\,125$ viewers nationwide.

- c) $\frac{328\,125}{1\,700\,000} \approx 0.193$, so about 19.3% of the viewers watched the children's program.

- d) $\frac{504}{3200} \cdot 15\,000\,000 = 2\,362\,500$, and $15\,000\,000 \cdot 0.6 = 9\,000\,000$, so

$$\frac{2\,362\,500}{9\,000\,000} = 0.2625 = 26.25\%$$

This is more than 25%, so the rating of the show is good.

8. Let A be the interval $\left(-\frac{9\pi}{4}, 2\pi\right]$ and B the interval $\left[\frac{\pi}{6}, \frac{11\pi}{2}\right)$. Find the solutions of the equation

$$(7 \sin x - 2)^2 = (5 \sin x - 6)^2 - 4(\sin x + 2)$$

on the sets $A \setminus B$ and $A \cap B$.

(16 points)

Solution: After expanding and substituting $\sin x = y$:

$$49y^2 - 28y + 4 = 25y^2 - 60y + 36 - 4y - 8, \quad \text{that is,} \quad 2y^2 + 3y - 2 = 0$$

so $y_1 = \frac{1}{2}$ and $y_2 = -2$. y_2 is not possible, since $-1 \leq \sin x \leq 1$. So $\sin x = \frac{1}{2}$, that is, $x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$.

$A \setminus B = \left(-\frac{9\pi}{4}, \frac{\pi}{6}\right)$; the roots in this interval are $-\frac{11\pi}{6}$ and $-\frac{7\pi}{6}$.

$A \cap B = \left[\frac{\pi}{6}, 2\pi\right]$; the roots in this interval are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Note: KoMaL writes $A \setminus B$ as closed at $\frac{\pi}{6}$; since $\frac{\pi}{6} \in B$, the interval is open there. The roots are not affected.

9. Solve the following equation.

$$x + \log_{21}(3^x + 1) = x \cdot \log_{21} 7 + \log_{21} 756$$

(16 points)

Solution: By the rules of logarithms,

$$\begin{aligned} \log_{21} 21^x + \log_{21}(3^x + 1) &= \log_{21} 7^x + \log_{21} 756 \\ \log_{21} 21^x (3^x + 1) &= \log_{21} (7^x \cdot 756) \end{aligned}$$

Since the logarithm function is strictly monotonic, $21^x (3^x + 1) = 7^x \cdot 756$, that is, $7^x 3^x (3^x + 1) = 7^x \cdot 756$. Since $7^x \neq 0$, we can divide by it:

$$3^x (3^x + 1) = 756, \quad \text{that is,} \quad (3^x)^2 + 3^x - 756 = 0$$

$$3^x = \frac{-1 \pm \sqrt{1 + 4 \cdot 756}}{2} = \frac{-1 \pm 55}{2}, \quad \text{so} \quad 3^x = 27 \quad \text{or} \quad 3^x = -28$$

-28 is not possible, since $3^x > 0$. So $3^x = 3^3$, and $x = 3$.