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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. A cyclist rode the first half of his trip at 12 km/h and the second half at 18 km/h. On the way back he would like to ride at a constant speed and cover the distance in the same time as on the way there. What should his speed be?

(11 points)

Solution: Let the length of the trip be $2s$ km. The cyclist rode the first half in $\frac{s}{12}$ hours and the second half in $\frac{s}{18}$ hours. He wants to ride back, that is, the $2s$ km, in $\frac{s}{12} + \frac{s}{18}$ hours, so his speed should be

$$\frac{2s}{\frac{s}{12} + \frac{s}{18}} = \frac{2}{\frac{1}{12} + \frac{1}{18}} = 14.4 \text{ km/h}$$

2. Solve the following system of equations over the pairs of integers.

$$\begin{aligned} x^2 - 4y^2 - 7x + 14y &= 0 \\ xy + x + 2y &= 0 \end{aligned}$$

(13 points)

Solution: The first equation can be factored: $(x - 2y)(x + 2y - 7) = 0$. So $x = 2y$ or $x = 7 - 2y$. In both cases we substitute the expression for x into the second equation.

If $x = 2y$: $2y^2 + 4y = 0$, so $y_1 = 0$, $x_1 = 0$ and $y_2 = -2$, $x_2 = -4$.

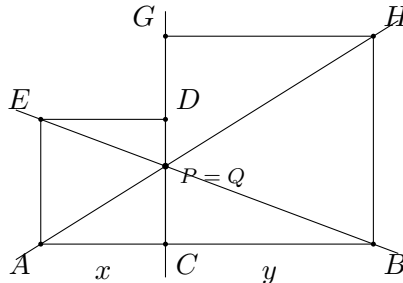
If $x = 7 - 2y$: $2y^2 - 7y - 7 = 0$, which has no integer solutions.

The solutions of the system are $(0, 0)$ and $(-4, -2)$.

3. We choose a point C on the segment AB such that $AC = x$ and $BC = y$. On the same side of the segment AB we draw the squares $ACDE$ and $CBHG$. The line DC meets the line BE at P , and the line AH at Q . Express the lengths CQ and CP in terms of x and y .

(13 points)

Solution:



The triangles PCB and EAB are similar (they have a right angle and a common angle at B), so

$$\frac{PC}{CB} = \frac{EA}{AB}, \quad \text{that is,} \quad \frac{PC}{y} = \frac{x}{x+y}, \quad PC = \frac{xy}{x+y}$$

In the same way the triangles QCA and HBA are similar, so $\frac{QC}{CA} = \frac{HB}{BA}$, that is, $\frac{QC}{x} = \frac{y}{x+y}$, and $QC = \frac{xy}{x+y}$.

So $CP = CQ = \frac{xy}{x+y}$; it also follows that $P = Q$.

4. How many terms of the sequence

$$a_n = -1 + \lg(n+3)$$

are greater than 1 but less than 2?

(14 points)

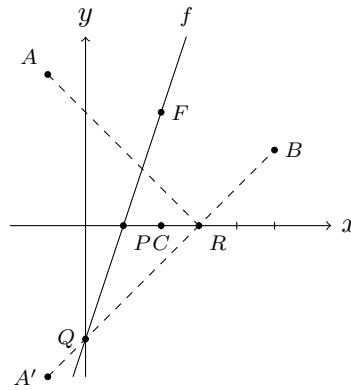
Solution: We need the number of positive integer solutions of $1 < -1 + \lg(n+3) < 2$, that is, $2 < \lg(n+3) < 3$. Using logarithms, $\lg 100 < \lg(n+3) < \lg 1000$, and since $\lg$ is increasing, $100 < n+3 < 1000$, that is, $97 < n < 997$. The suitable values of n are 98, 99, ..., 996, so 899 terms of the sequence are between 1 and 2.

Part II

5. In a coordinate system the points $A(-1, 4)$ and $B(5, 2)$ are given. Find

- the point P on the x -axis with $PA = PB$, and the point Q on the y -axis with $QA = QB$; (5 points)
- the point R on the x -axis for which $AR + RB$ is minimal; (5 points)
- the point C on the x -axis for which $AC^2 + CB^2$ is minimal. (6 points)

Solution: a) Every point of the perpendicular bisector f of AB is the same distance from A and B , so we need the intersection points of this line with the axes. The midpoint of AB is $F(2, 3)$, and a normal vector of f is $(3, -1)$, so f : $3x - y = 3$. It meets the x -axis at $P(1, 0)$ and the y -axis at $Q(0, -3)$.



b) The reflection of $A(-1, 4)$ in the x -axis is $A'(-1, -4)$. The line $A'B$ is $x - y = 3$. By the properties of reflection, $AR + RB$ is minimal where this line meets the x -axis: $R(3, 0)$.

c) Let $C(c, 0)$. We minimize

$$(c+1)^2 + 4^2 + (c-5)^2 + 2^2 = 2c^2 - 8c + 46 = 2(c-2)^2 + 38$$

which is minimal at $c = 2$. The point is $C(2, 0)$.

6. a) We roll two fair dice, a yellow one and an orange one. Let c be the number on the yellow die and n the number on the orange die. What is the probability that the equation $cx^2 = n$ has two integer roots? (8 points)

b) We roll three fair dice. Let p be the number on the red die, f the number on the white die, and z the number on the green die. What is the probability that the equation $px^2 + fx + z = 0$ has two different roots that are reciprocals of each other? (8 points)

Solution: a) $\frac{n}{c}$ must be a perfect square. Since the numbers rolled are $1, 2, \dots, 6$, the possible values of the quotient are 1 and 4. The favorable cases are $c = n$ (six cases) and $c = 1, n = 4$:

c	1	2	3	4	5	6	1
n	1	2	3	4	5	6	4

There are 36 cases in all and 7 favorable cases, so the probability is $\frac{7}{36}$.

b) If the two roots are reciprocals, their product is 1, that is, $\frac{z}{p} = 1, p = z$. For two different roots, the discriminant must be positive: $f^2 - 4pz > 0$, that is, $f > 2p$. If $p = z = 1$, then f can be 3, 4, 5, 6; if $p = z = 2$, then f can be 5, 6; for larger $p = z$ there is no suitable f . So there are 6 good triples out of 6^3 , and the probability is

$$\frac{6}{6^3} = \frac{1}{36}$$

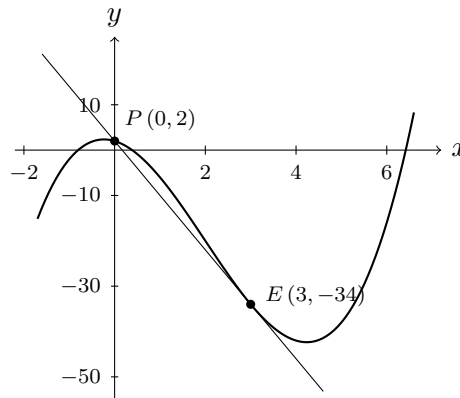
7. a) Show that every member of the family of curves $y = x^3 + (a - 3)x^2 - 3x + 2$ passes through one point. Find the coordinates of this fixed point. (6 points)
- b) How should the parameter a be chosen so that the tangent line of the corresponding curve at $x_0 = 3$ passes through the point $(0, 2)$? (10 points)

Solution: a) The parameter appears only in the term ax^2 , so the value of y does not depend on a exactly when $x = 0$. Then $y = 2$ for every a , so every curve passes through $P(0, 2)$. (For example, the curves for $a = 0$ and $a = 1$ meet only at this point.)

b) The slope of the tangent line at $x_0 = 3$ is given by the derivative:

$$f'(x) = 3x^2 + 2(a - 3)x - 3, \quad f'(3) = 6a + 6$$

The point of tangency is $E(3, 9a - 7)$, so the tangent line is $y - 9a + 7 = (6a + 6)(x - 3)$. It passes through $(0, 2)$ if $9 - 9a = -18a - 18$, that is, $a = -3$. Then the curve is $y = x^3 - 6x^2 - 3x + 2$, the point of tangency is $E(3, -34)$, and the tangent line is $y = -12x + 2$.



8. The inscribed circle of a right triangle has radius 2, and the radii of the excircles tangent to the legs are 3 and 10. What is the radius of the excircle tangent to the hypotenuse? (16 points)

Solution: With the usual notation, $\varrho_a = s - b$ and $\varrho_b = s - a$, where s is the semiperimeter. Adding the two equations, $\varrho_a + \varrho_b = 2s - a - b = c$, so the hypotenuse is $c = 3 + 10 = 13$.

For a right triangle the excircle tangent to the hypotenuse has radius $\varrho_c = s$. The inscribed circle touches the hypotenuse; let x be the distance of the point of tangency on BC from B . The legs are $x + 2$ and $(13 - x) + 2 = 15 - x$, so by the Pythagorean theorem

$$(x + 2)^2 + (15 - x)^2 = 169, \quad \text{that is,} \quad x^2 - 13x + 30 = 0$$

so $x = 10$ or $x = 3$. The legs are $a = 5$ and $b = 12$, and

$$\varrho_c = s = \frac{5 + 12 + 13}{2} = 15$$

Remark: In every right triangle, $\varrho_a + \varrho_b + \varrho = \varrho_c$.

9. The 12 lattice points (points with integer coordinates) on the circle of radius 13 centered at the origin determine a 12-gon.
- a) Is this 12-gon a tangential polygon (does it have an inscribed circle)? (5 points)
- b) Find the area of the 12-gon. (7 points)
- c) This 12-gon is the base of a right pyramid of height 13. What angles do the lateral faces make with the base? (4 points)

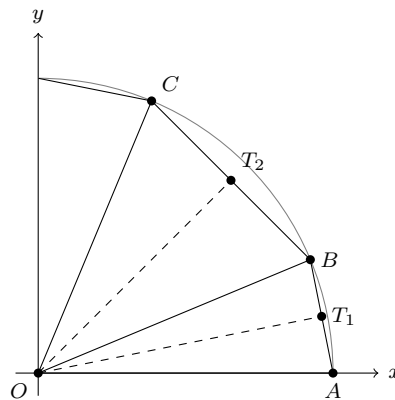
Solution: a) The 12 lattice points are $(\pm 13, 0)$, $(0, \pm 13)$, $(\pm 5, \pm 12)$, $(\pm 12, \pm 5)$. Three neighboring vertices are $A(13, 0)$, $B(12, 5)$, $C(5, 12)$. If the 12-gon had an inscribed circle, its center would be the origin, so it would have to touch both $AB = \sqrt{26}$ and $BC = 7\sqrt{2}$. In the isosceles triangle OAB the altitude from O is

$$OT_1 = \sqrt{13^2 - \left(\frac{\sqrt{26}}{2}\right)^2} = 5\sqrt{\frac{13}{2}}$$

and in the isosceles triangle OBC it is

$$OT_2 = \sqrt{13^2 - \left(\frac{7\sqrt{2}}{2}\right)^2} = \frac{17\sqrt{2}}{2}$$

Since $OT_1 \neq OT_2$, the 12-gon has no inscribed circle: it is not a tangential polygon.



- b) The 12-gon consists of 8 triangles congruent to OAB and 4 triangles congruent to OBC :

$$t_{OAB} = \frac{\sqrt{26} \cdot 5\sqrt{\frac{13}{2}}}{2} = \frac{65}{2}, \quad t_{OBC} = \frac{7\sqrt{2} \cdot \frac{17\sqrt{2}}{2}}{2} = \frac{119}{2}$$

so the area is $T = 8 \cdot \frac{65}{2} + 4 \cdot \frac{119}{2} = 498$.

- c) Let P be the apex. There are only two kinds of lateral faces; the angles come from the right triangles POT_1 and POT_2 :

$$\tan \alpha_1 = \frac{13}{5\sqrt{\frac{13}{2}}} \approx 1.0198, \quad \alpha_1 \approx 45.56^\circ; \quad \tan \alpha_2 = \frac{13}{\frac{17\sqrt{2}}{2}} \approx 1.0815, \quad \alpha_2 \approx 47.24^\circ$$