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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following inequality over the real numbers.

$$27^x < 6 \cdot 9^x + 3^{x+3}$$

(12 points)

Solution: Substituting $a = 3^x$, the inequality becomes $a^3 - 6a^2 - 27a < 0$. We can divide both sides by a , since $3^x > 0$ for every x : $a^2 - 6a - 27 < 0$, that is, $(a - 9)(a + 3) < 0$. Its solution is $-3 < a < 9$, that is, $-3 < 3^x < 3^2$. Since $3^x > 0$ for every x , the solution is $x < 2$.

2. The longest side of a triangle is 16 and the shortest side is 10. The largest angle of the triangle is twice the smallest angle. Find the angles and the missing side of the triangle.

(12 points)

Solution: The largest angle is opposite the longest side and the smallest angle opposite the shortest side; so with $a = 16$ and $b = 10$, $\alpha = 2\beta$. By the Law of Sines,

$$\frac{16}{10} = \frac{\sin 2\beta}{\sin \beta}$$

Using $\sin 2\beta = 2 \sin \beta \cos \beta$, we get $1.6 = 2 \cos \beta$, that is, $\cos \beta = 0.8$. So $\beta \approx 36.87^\circ$, $\alpha \approx 73.74^\circ$, and $\gamma = 180^\circ - \alpha - \beta \approx 69.39^\circ$; these satisfy $\beta < \gamma < \alpha$. By the Law of Sines, the missing side is $c = \frac{10 \sin \gamma}{\sin \beta} = 15.6$.

3. Anna and Bence roll four fair dice in a game. Anna wins if there are equal numbers among those rolled, and Bence wins if there is at least one 6 among them. If both conditions hold, the game is a draw; if neither holds, they keep playing.

a) Which of them has the greater chance of winning?

(7 points)

b) What is the probability that a game ends in a draw?

(6 points)

Solution: a) There are $6^4 = 1296$ outcomes. All four numbers are different in $6 \cdot 5 \cdot 4 \cdot 3 = 360$ outcomes, so there are equal numbers in $1296 - 360 = 936$ outcomes. There is no 6 in $5^4 = 625$ outcomes, so there is at least one 6 in $1296 - 625 = 671$ outcomes.

Let a be the number of outcomes in which Anna wins, b in which Bence wins, d the number of draws, and c the number in which neither condition holds (all different and no 6): $c = 5 \cdot 4 \cdot 3 \cdot 2 = 120$. Then $a + b + c + d = 1296$, $a + d = 936$, and $b + d = 671$, so $d - c = 311$, and $d = 431$. So Anna wins in $a = 936 - 431 = 505$ outcomes and Bence in $b = 671 - 431 = 240$ outcomes: Anna has the greater chance of winning.

b) The probability of a draw is

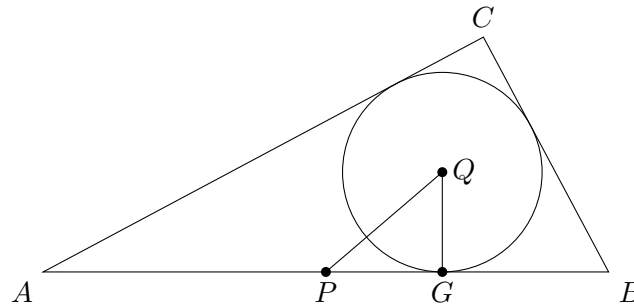
$$p = \frac{431}{1296} \approx 0.33$$

(This is the probability for one roll; since a roll with neither condition is repeated, the probability that a game ends in a draw is $\frac{431}{1296-120} = \frac{431}{1176} \approx 0.37$.)

4. The sides of the triangle ABC are $AB = 34$, $BC = 16$, $AC = 30$. Let P be the center of the circumscribed circle and Q the center of the inscribed circle. Find the length of the segment PQ .

(14 points)

Solution: Since $16^2 + 30^2 = 34^2$, the triangle has a right angle at C , so P is the midpoint of the hypotenuse AB .



Let G be the point where the inscribed circle touches AB , and ρ its radius. The area of the right triangle is $t = \frac{16 \cdot 30}{2} = 240$, and the semiperimeter is $s = 40$, so $\rho = \frac{t}{s} = 6$. The tangent segments from C are equal to ρ (the inscribed circle, C , and the points of tangency on the legs form a square), so the tangent segment from B is $16 - 6 = 10$: $BG = 10$, and $PG = 17 - 10 = 7$. In the right triangle PGQ ,

$$PQ = \sqrt{7^2 + 6^2} = \sqrt{85} \approx 9.22$$

Part II

5. Solve the following equation over the pairs of real numbers.

$$\sqrt{4 - 4x^2 + 4xy - y^2} = \log_x (y - 2) + \log_{y-2} x$$

(16 points)

Solution: Writing the expression under the square root as $4 - (2x - y)^2$, we see that $4 - (2x - y)^2 \leq 4$, since $(2x - y)^2 \geq 0$; so the left side is at most 2 wherever it is defined.

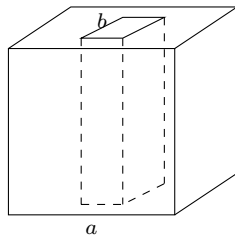
Since $\log_a b = \frac{1}{\log_b a}$, the right side is the sum of a number and its reciprocal. For a positive number this sum is at least 2 (with equality only if the number is 1), and for a negative number it is at most -2 , which is impossible here, since the left side is not negative.

So equality can hold only if both sides are exactly 2: $2x - y = 0$, that is, $y = 2x$, and $\log_x (y - 2) = 1$, that is, $x = y - 2$. The solution of this system is $x = 2$, $y = 4$, which is indeed a solution of the original equation.

6. We drill a hole through a cube of edge a , perpendicular to one of its faces. The removed part is a square prism whose base is a square of side b and whose height is a .
- a) Prove that if the volume of the resulting solid is 25% less than the volume of the original cube, then its surface area is 25% more than the surface area of the original cube.
- b) Prove that no other number can be written in place of 25 in this statement, that is, a $k\%$ decrease in volume cannot come with a $k\%$ increase in surface area for any a and b , if $k \neq 25$.

(16 points)

Solution: a) The volume of the cube is a^3 , and the volume of the square prism is b^2a . By the condition, $b^2a = \frac{1}{4}a^3$, so $b^2 = \frac{1}{4}a^2$, that is, $a = 2b$.



The surface area of the original cube is $6a^2$; after drilling, the surface area of the solid is $6a^2 - 2b^2 + 4ab$. Substituting $a = 2b$, these are $24b^2$ and $24b^2 - 2b^2 + 8b^2 = 30b^2$. Since $30b^2 = 1.25 \cdot 24b^2$, the statement is proved.

b) If the volume decreases by $k\%$, then $b^2a = \frac{k}{100}a^3$, so $a = \frac{10b}{\sqrt{k}}$. If the surface area increases by $k\%$, then

$$-2b^2 + 4ab = \frac{6a^2k}{100}$$

Substituting $a = \frac{10b}{\sqrt{k}}$: $-2b^2 + \frac{40b^2}{\sqrt{k}} = 6b^2$, that is, $\frac{40}{\sqrt{k}} = 8$, so $k = 25$.

7. 30 students of a class wrote a math test. The mode of the grades is 2, the median 3.5, the range 4, and the average 3.2. We also know that there was one more 4 than 5, and the standard deviation of the grades is less than 1.36. Find the frequencies of the grades. (The grades are 1, 2, 3, 4, 5.)

(16 points)

Solution: From the median 3.5, the two middle grades in increasing order are 3 and 4 (2 and 5 are not possible, since then the median would not be 3.5). So there are 15 grades of 4 and 5, and since there is one more 4 than 5, there are 8 fours and 7 fives.

Since the average is 3.2, the sum of the grades is $30 \cdot 3.2 = 96$; the fours and fives add up to $8 \cdot 4 + 7 \cdot 5 = 67$, so the 1's, 2's, and 3's add up to 29. Fifteen 2's would add up to 30, so there is one more 1 than 3 among these grades; and since 2 is the mode, there are at least 9 twos. The range is 4, so there is at least one 1. This leaves two possibilities:

I. 10 twos, 2 threes, and 3 ones; II. 12 twos, 1 three, and 2 ones.

(14 twos, no threes, and 1 one is not possible, since there is a 3 among the two middle grades.) Computing the standard deviation in the two cases, only case II gives less than 1.36 (about 1.352; case I gives about 1.376).

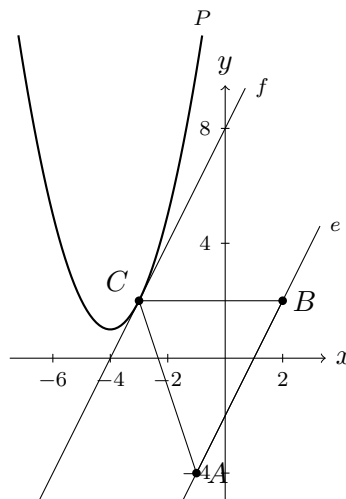
The frequencies are: 2 ones, 12 twos, 1 three, 8 fours, and 7 fives.

8. The equation of the line e is $2x - y = 2$, and the equation of the parabola P is $y = x^2 + 8x + 17$. The points A and B lie on the line e , and the point C on the parabola P . The x -coordinate of A is -1 , and the y -coordinate of B is 2 .

Find the coordinates of C so that the area of the triangle ABC is as small as possible. What is this minimal area?

(16 points)

Solution: Substituting into the equation of e , we get $A(-1, -4)$ and $B(2, 2)$. The area of the triangle ABC is smallest when the altitude to the side AB is smallest, so we need the point of the parabola closest to the line e . Let f be the line parallel to e that is tangent to the parabola; then C is the point of tangency.



Since $e: y = 2x - 2$, the line f is $y = 2x + c$. Substituting into the equation of P : $x^2 + 6x + 17 - c = 0$. The line f is tangent to the parabola exactly when the discriminant is 0: $36 - 68 + 4c = 0$, so $c = 8$. Then $x = -3$, and $C(-3, 2)$.

Since $BC = 5$ (both points have y -coordinate 2) and the altitude to the side BC is 6, the area of the triangle ABC is $\frac{5 \cdot 6}{2} = 15$.

9. Three two-digit natural numbers are three consecutive terms of an increasing arithmetic sequence, and their sum is 198. Writing the three numbers one after the other in increasing order, the resulting 6-digit number is divisible by 2009. Find the three numbers.

(16 points)

Solution: The middle number is $198 \div 3 = 66$, the first is $66 - d$, and the third is $66 + d$ ($d > 0$, $d \in \mathbb{Z}$). The 6-digit number formed from them is

$$10\,000(66 - d) + 100 \cdot 66 + 66 + d = 666\,666 - 9999d = 99(6734 - 101d)$$

Since $2009 = 7^2 \cdot 41$ and $99 = 3^2 \cdot 11$ has no common factor with it, $2009 \mid 6734 - 101d$. Since $d > 0$ (and $66 - d$ is a two-digit number), $6734 - 101d$ can only be 2009, $2 \cdot 2009$, or $3 \cdot 2009$. In the first two cases d is not an integer, and in the third case $d = 7$.

So the three numbers are 59, 66, 73, and the 6-digit number is $596\,673 = 2009 \cdot 297$.