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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

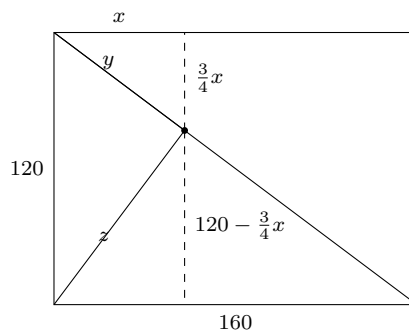
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The sides of a rectangular fish pond are 120 m and 160 m long. A pole was driven in along one diagonal of the rectangle, at a point from which the two endpoints of a shorter side are seen at a right angle. How far is the pole from the sides of the pond?

(11 points)

Solution: Let the distance of the pole from the nearer shorter side be x . By similarity (the diagonal has slope $\frac{120}{160} = \frac{3}{4}$), the distance of the pole from the nearer longer side is $\frac{3}{4}x$. Let y and z be the distances of the pole from the two endpoints of the shorter side.



By the Pythagorean theorem in three right triangles:

$$y^2 + z^2 = 120^2, \quad y^2 = x^2 + \left(\frac{3}{4}x\right)^2, \quad z^2 = x^2 + \left(120 - \frac{3}{4}x\right)^2$$

Substituting the last two into the first: $\frac{25}{8}x^2 - 180x = 0$, so $x = 0$ or $x = 57.6$; x cannot be 0. So the pole is 57.6 m and $160 - 57.6 = 102.4$ m from the shorter sides, and $\frac{3}{4} \cdot 57.6 = 43.2$ m and $120 - 43.2 = 76.8$ m from the longer sides.

Note: KoMaL gives 62.4 m and 116.8 m for the other two distances, subtracting from the wrong side lengths; the correct values are 102.4 m and 76.8 m.

2. Solve the following inequalities over the real numbers.

a) $\sqrt{x^2 - 5x + 4} > -1$

b) $\log_{\frac{1}{2}}(4^x - 5 \cdot 2^x + 8) < -2$

(12 points)

Solution: a) The square root is never negative, so the inequality holds on its whole domain, that is, where $x^2 - 5x + 4 \geq 0$. The solution is $x \leq 1$ or $x \geq 4$.

b) The base $\frac{1}{2}$ logarithm function is strictly decreasing, so the inequality is $4^x - 5 \cdot 2^x + 8 > 4$ (and then the logarithm is defined). This is a quadratic inequality in 2^x : $(2^x)^2 - 5 \cdot 2^x + 4 > 0$, so $2^x > 4$, that is, $x > 2$, or $2^x < 1$, that is, $x < 0$. The solution is $x > 2$ or $x < 0$.

3. The chess players of two schools played a tournament. Everyone played everyone else once. First the games within each school were played, 36 games in all. When the students of the two schools played each other, there were 42 games. How many students from each school took part in the tournament?

(14 points)

Solution: Let the numbers of players of the two schools be x and y . Then

$$\frac{x(x-1)}{2} + \frac{y(y-1)}{2} = 36$$

is the number of games within the schools, and $xy = 42$ is the number of games between the two schools. We need positive integers, so only the pairs 1–42, 2–21, 3–14, 6–7 are possible. Writing the first equation as $x^2 + y^2 - (x + y) = 72$, only $x = 6$, $y = 7$ (or the other way around) works. So 6 students from one school and 7 from the other took part in the tournament.

4. At the beginning of 2005 Antal deposited 100 000 Ft in a bank at 20% annual interest. Starting in 2005, at the beginning of every year Béla deposited b forints, also at 20% annual interest. At the end of 2009, Antal's and Béla's deposits had grown to the same value (from 2005 to 2009 neither of them took any money out). What is the value of b , rounded to 1000 Ft?

(14 points)

Solution: Antal's money grows by 20% every year, so in 5 years it grows to $100\,000 \cdot 1.2^5 = 248\,832$ Ft. The b forints Béla deposits at the beginning of each year earn interest for 5, 4, 3, 2, 1 years, so his money grows to

$$b \cdot 1.2^5 + b \cdot 1.2^4 + b \cdot 1.2^3 + b \cdot 1.2^2 + b \cdot 1.2 = b \cdot 8.929\,92$$

Since the two deposits are equal, $248\,832 = b \cdot 8.929\,92$, so $b \approx 27\,865$, that is, $b \approx 28\,000$ Ft.

Part II

5. The axis of symmetry of the kite $ABCD$ is the diagonal AC , where $A(0, 0)$ and $C(8, 10)$. The area of the kite is 41 square units. One diagonal divides the other in the ratio 3 : 2, counting from the origin. Find the coordinates of the missing vertices.

(16 points)

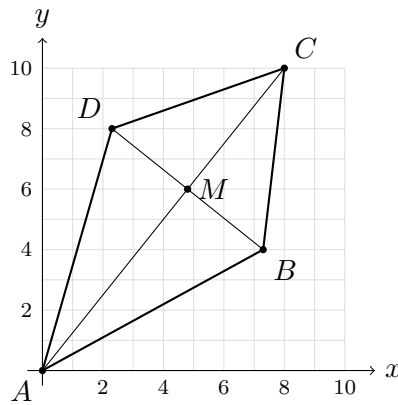
Solution: The length of the diagonal AC is $\sqrt{8^2 + 10^2} = \sqrt{164}$. Since $T = \frac{AC \cdot BD}{2}$, $BD = \frac{82}{\sqrt{164}} = \sqrt{41}$.

The intersection point M of the diagonals divides AC in the ratio 3 : 2, so $M(\frac{24}{5}, 6)$. The line BD is perpendicular to AC at M :

$$4x + 5y = \frac{246}{5}$$

The circle around M with radius $MD = \frac{BD}{2} = \frac{\sqrt{41}}{2}$ is $(x - \frac{24}{5})^2 + (y - 6)^2 = \frac{41}{4}$. The intersection points of the line and the circle are the vertices B and D :

$$B(7.3, 4), \quad D(2.3, 8)$$



6. We rotate an isosceles triangle about one of its legs, and then about its base. Let V_1 and V_2 denote the volumes of the resulting solids of revolution. Find the angles of the triangle if $V_1 : V_2 = 3 : 7$.

(16 points)

Solution: Let the base be $BC = a$, the legs b , and the altitudes to them m_a and m_b . Rotating the triangle about the leg AC , the volume of the solid is $V_1 = \frac{m_b^2 \pi \cdot b}{3}$, and rotating it about the base, $V_2 = \frac{m_a^2 \pi \cdot a}{3}$. So

$$\frac{3}{7} = \frac{V_1}{V_2} = \frac{m_b^2 b}{m_a^2 a}$$

Computing the area in two ways, $2T = am_a = bm_b$, so $\frac{m_b}{m_a} = \frac{a}{b}$, and

$$\frac{3}{7} = \frac{a^2 b}{b^2 a} = \frac{a}{b}$$

For the base angle β , $\cos \beta = \frac{a/2}{b} = \frac{3}{14}$, so $\beta \approx 77.6^\circ$. The angles of the triangle are 77.6° , 77.6° , and 24.8° .

7. a) Find the digits a, b, c for which the one-digit $\overline{a}$, the two-digit $\overline{ba}$, and the three-digit $\overline{cba}$ positive numbers are consecutive terms of a geometric sequence.
 b) Find the digits a, b, c for which the one-digit $\overline{a}$, twice the two-digit $\overline{ba}$, and the three-digit $\overline{cba}$ positive numbers are consecutive terms of an arithmetic sequence.

(16 points)

Solution: a) For consecutive terms of a geometric sequence, $(10b + a)^2 = a(100c + 10b + a)$, so $ab = 10(ac - b^2)$, and $10 \mid ab$. So either $a = 5$ and b is 2, 4, 6, or 8, or $b = 5$ and a is 2, 4, 6, or 8. Only the first case gives a solution: $a = 5, b = 2, c = 1$. Indeed 5, 25, 125 are consecutive terms of a geometric sequence.

b) For consecutive terms of an arithmetic sequence,

$$\frac{a + (100c + 10b + a)}{2} = 2(10b + a), \quad \text{which gives} \quad 50c = 15b + a$$

The left side is divisible by 5, so a is also divisible by 5, and positive: $a = 5$. Then $10c = 3b + 1$. The right side is less than 30, so c is 1 or 2. If $c = 1$, then $b = 3$; if $c = 2$, there is no suitable b . The solution is $a = 5, b = 3, c = 1$: indeed 5, $2 \cdot 35 = 70$, 135 are consecutive terms of an arithmetic sequence.

8. A shooter hits the target with probability $\frac{1}{4}$.

a) What is the probability that out of 7 shots at least 2 hit the target?

b) At least how many shots does he have to take to hit the target with probability greater than $\frac{2}{3}$?

(16 points)

Solution: a)

$$p(\text{at least 2}) = 1 - p(0 \text{ hits}) - p(1 \text{ hit}) = 1 - \left(\frac{3}{4}\right)^7 - \binom{7}{1} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^6 \approx 0.555$$

b) The probability of missing all of n shots is $\left(\frac{3}{4}\right)^n$; we need $\left(\frac{3}{4}\right)^n < \frac{1}{3}$, that is,

$$n \lg \frac{3}{4} < \lg \frac{1}{3}, \quad \text{so} \quad n > \frac{\lg \frac{1}{3}}{\lg \frac{3}{4}} \approx 3.82$$

If the shooter takes at least 4 shots, he hits the target with probability greater than $\frac{2}{3}$.

9. Graph the function

$$f(x) = 3^{1+\log_3[\cos(x+\frac{\pi}{4})]}$$

on the interval $\left(-\frac{3\pi}{4}, \frac{9\pi}{4}\right]$. Find the domain, the range, the zeros, the intervals of increase and decrease, and the period of f .

(16 points)

Solution: Only positive numbers have a logarithm, so $\cos(x + \frac{\pi}{4}) > 0$, that is, $-\frac{\pi}{2} + 2k\pi < x + \frac{\pi}{4} < \frac{\pi}{2} + 2k\pi$. So the domain of the function is

$$D_f = \left\{ x \in \mathbb{R} \mid -\frac{3\pi}{4} + 2k\pi < x < \frac{\pi}{4} + 2k\pi, k \in \mathbb{Z} \right\}$$

On the domain, $f(x) = 3 \cdot \cos(x + \frac{\pi}{4})$.

The range of the function is $(0, 3]$; it has no zeros and no minimum; its maximum is 3, at $x = -\frac{\pi}{4} + 2l\pi, l \in \mathbb{Z}$; its period is 2π . It is strictly increasing on $(-\frac{3\pi}{4} + 2n\pi, -\frac{\pi}{4} + 2n\pi]$ and strictly decreasing on $[-\frac{\pi}{4} + 2m\pi, \frac{\pi}{4} + 2m\pi)$, where $n, m \in \mathbb{Z}$.

