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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Let A be the domain of the expression a), and B the domain of the expression b).

$$\text{a) } \log_{x+5}(x^2 - 9x + 14) \qquad \text{b) } \sqrt{\frac{10-x}{x^2+4}}$$

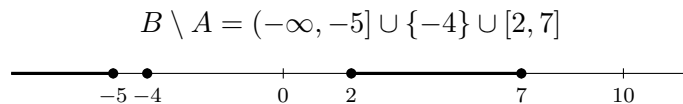
Show the elements of the set $B \setminus A$ on the number line.

(11 points)

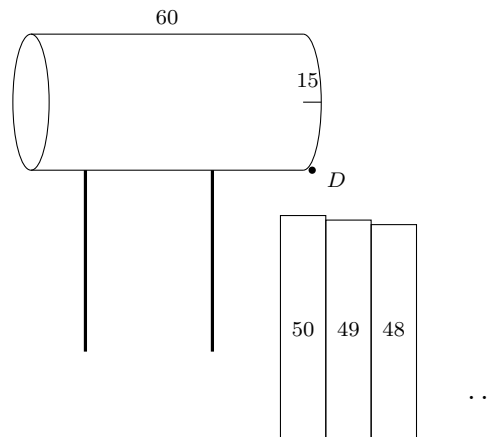
Solution: For the expression a), $x+5 > 0$, $x+5 \neq 1$, and $x^2 - 9x + 14 > 0$. The first two conditions give $x > -5$ and $x \neq -4$. The zeros of the quadratic are 2 and 7, so $x < 2$ or $x > 7$. Altogether, $A = (-5, -4) \cup (-4, 2) \cup (7, \infty)$.

For the expression b), $\frac{10-x}{x^2+4} \geq 0$. The denominator is positive for every x , so this holds exactly when $x \leq 10$: $B = (-\infty, 10]$.

The elements of $B \setminus A$ are the elements of B not in A :



2. A closed container in the shape of a right circular cylinder has a base radius of 15 cm and a height of 60 cm. The container is fixed horizontally on two rods. It is half full of water, and at its bottom there is a plug D . If the plug is pulled out, the water flows into open-top containers in the shape of right prisms placed under the tank. The base of each prism is a square of side 10 cm. The first prism is 50 cm high, the second 49 cm, the third 48 cm, and so on. When the first prism is full, the water flows over into the second; if that is full, it flows over into the third, and so on.



How many prisms will be filled with water?

(12 points)

Solution: The volume of the water is half the volume of the container:

$$V = \frac{r^2 \pi m}{2} = \frac{15^2 \pi \cdot 60}{2} \approx 21\,205.75 \text{ cm}^3$$

The volumes of the containers are $V_{50} = 5000$, $V_{49} = 4900$, $V_{48} = 4800$, $V_{47} = 4700$, $V_{46} = 4600$, ... Since

$$5000 + 4900 + 4800 + 4700 = 19\,400 \quad \text{and} \quad 5000 + 4900 + 4800 + 4700 + 4600 = 24\,000$$

4 containers (the ones 50, 49, 48, and 47 cm high) will be filled with water.

3. We put the perfect squares into “packages” in the following way:

package 1: (1), package 2: (4, 9), package 3: (16, 25, 36), package 4: (49, 64, 81, 100), etc.

a) Which number does package 16 start with?

b) Which package contains the perfect square 8281?

(14 points)

Solution: a) Package n contains n perfect squares. So the first number of package 16 is the perfect square after all the squares in the first 15 packages, which contain

$$1 + 2 + \dots + 15 = \frac{(1 + 15) \cdot 15}{2} = 120$$

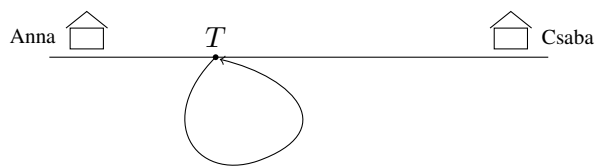
numbers. So package 16 starts with the 121st perfect square, $121^2 = 14\,641$.

b) The first $k - 1$ packages contain $\frac{k(k-1)}{2}$ numbers, so package k starts with the $\left(\frac{k(k-1)}{2} + 1\right)$ st perfect square, and package $k + 1$ with the $\left(\frac{k(k+1)}{2} + 1\right)$ st. Since $8281 = 91^2$, it is in package k if

$$\frac{k(k-1)}{2} + 1 \leq 91 < \frac{k(k+1)}{2} + 1, \quad \text{that is,} \quad k(k-1) \leq 180 < k(k+1)$$

Trying a few values, $13 \cdot 12 = 156 \leq 180 < 13 \cdot 14 = 182$, so $k = 13$: 8281 is in package 13. (Package 13 starts with the 79th square, so 8281, the 91st square, is the last number of package 13.)

4. Anna’s and Csaba’s houses are on a straight road. They agreed to go cycling together. At noon they met at a point T of the road between their houses, where a bicycle loop starts.



When they met, Anna’s daily average speed meter showed 23 km/h, and Csaba’s showed 28 km/h. (Until noon, both of them had used their bikes only for this trip.) They rode the 26 km loop in 1 hour; then, back at the noon meeting point, they said goodbye and both went home. When they said goodbye, Anna’s average speed meter showed 25 km/h, and Csaba’s 27 km/h. How far apart do Anna and Csaba live?

(14 points)

Solution: Let S_a and t_a be the distance Anna rode until the meeting and the time it took. Then $S_a = 23t_a$. After the meeting they rode 26 km in one hour, so

$$\frac{S_a + 26}{t_a + 1} = 25, \quad \text{that is,} \quad 23t_a + 26 = 25t_a + 25$$

so $t_a = \frac{1}{2}$ and $S_a = 11.5$ km. In the same way, for Csaba $S_c = 28t_c$ and $S_c + 26 = 27(t_c + 1)$, so $28t_c + 26 = 27t_c + 27$, $t_c = 1$, and $S_c = 28$ km.

So Anna and Csaba live $11.5 + 28 = 39.5$ km apart.

Part II

5. In a lottery in a Swiss canton, 5 numbers have to be guessed out of the first 50 positive integers. A passionate player first chooses two numbers, then as the third number the sum of the first two, as the fourth number the sum of the first three, and finally as the fifth number the sum of the first four.

- At most how large can this player choose the smallest number?
- If he chooses the smallest number as large as possible, which numbers can appear on his ticket?
- What is the probability of hitting the jackpot if he fills out one ticket in every possible way according to this method?

(16 points)

Solution: a) Let the two chosen numbers be $a < b$. Then the numbers on the ticket are $a, b, a + b, 2a + 2b, 4a + 4b$. Since the largest number is at most 50, $4a + 4b \leq 50$, that is, $a + b \leq 12$. Since $a < b$, this is possible only if $a \leq 5$. So the smallest number is at most 5.

b) If $a = 5$, then $5 + b \leq 12$, so b is 6 or 7. The possible tickets are 5, 6, 11, 22, 44 and 5, 7, 12, 24, 48.

c) If $a = 5$, there are 2 tickets; if $a = 4$, b can be 5, 6, 7, 8: 4 tickets; if $a = 3$: 6 tickets; if $a = 2$: 8 tickets; if $a = 1$: 10 tickets. So there are $2 + 4 + 6 + 8 + 10 = 30$ such tickets. The number of all possible tickets is $\binom{50}{5} = 2\,118\,760$, so the probability of a jackpot is

$$\frac{30}{2\,118\,760} \approx 0.000\,014\,16$$

6. The function $f(x) = 2\sqrt{x}$ is defined on the interval $[0, 9]$.

- One vertex of an equilateral triangle is the origin, another vertex is on the x -axis, and the third vertex is on the graph of f . What is the area of this triangle?
- One side of a rectangle is on the x -axis, another side on the line $x = 9$, and one vertex on the graph of f . Find the largest possible area of such a rectangle.
- The line $x = a$ bisects the area between the graph of f and the x -axis. Find the value of the parameter a .

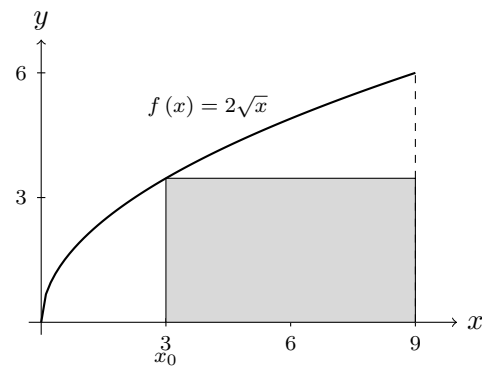
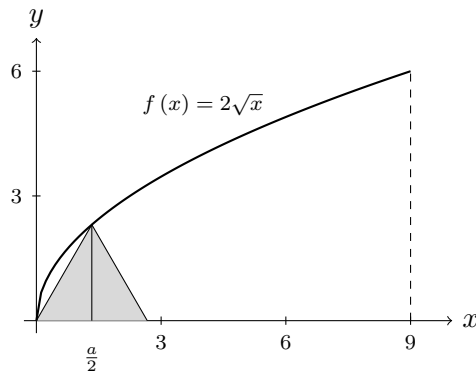
(16 points)

Solution: a) If the side of the equilateral triangle is a , then the value of the function at $x = \frac{a}{2}$ must equal the height of the triangle:

$$\frac{a\sqrt{3}}{2} = 2\sqrt{\frac{a}{2}}, \quad \text{that is,} \quad \frac{3a^2}{4} = \frac{4a}{2}$$

whose only positive solution is $a = \frac{8}{3}$. The area of the triangle is

$$T = \frac{a^2\sqrt{3}}{4} = \frac{16\sqrt{3}}{9} \approx 3.08$$



b) Let the left side of the rectangle be on the line $x = x_0$. The sides of the rectangle are $9 - x_0$ and $2\sqrt{x_0}$ ($0 < x_0 < 9$), and its area is

$$T(x_0) = 2\sqrt{x_0}(9 - x_0) = 18\sqrt{x_0} - 2\sqrt{x_0^3}$$

$$T'(x_0) = \frac{9}{\sqrt{x_0}} - 3\sqrt{x_0} = 0, \quad \text{so} \quad x_0 = 3$$

T' changes sign from positive to negative at $x_0 = 3$, so T has a maximum there. The largest area is $2\sqrt{3} \cdot (9 - 3) = 12\sqrt{3} \approx 20.78$.

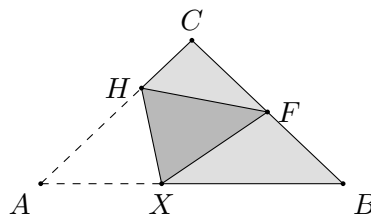
c) If the line $x = a$ bisects the area, then

$$2 \int_0^a 2\sqrt{x} \, dx = \int_0^9 2\sqrt{x} \, dx, \quad \text{that is,} \quad 2 \left[\frac{2}{3}\sqrt{x^3} \right]_0^a = \left[\frac{2}{3}\sqrt{x^3} \right]_0^9$$

$$\text{so } \frac{4}{3}\sqrt{a^3} = \frac{2}{3} \cdot 27, \quad \sqrt{a^3} = \frac{27}{2}, \quad \text{and}$$

$$a = \frac{9}{\sqrt[3]{4}} \approx 5.67$$

7. In a restaurant in Óbuda, the napkins in the shape of an isosceles triangle are folded so that the vertex A of the base AB of the triangle ABC goes to the midpoint F of the leg BC . Then one endpoint of the fold line goes to the point H of the leg AC that divides it into three equal parts, closer to C .



Where does the other endpoint X of the fold line go?

(16 points)

Solution: Let the base AB of the isosceles triangle ABC be 1, its legs b , and let $BX = x$. By the folding, $AX = XF = 1 - x$. Let the base angles of the triangle be α ; then $\angle ACB = 180^\circ - 2\alpha$.

By the Law of Cosines for the side HF of the triangle HFC , with $HC = \frac{1}{3}b$, $HF = AH = \frac{2}{3}b$, $FC = \frac{1}{2}b$:

$$\left(\frac{2}{3}b\right)^2 = \left(\frac{1}{3}b\right)^2 + \left(\frac{1}{2}b\right)^2 - 2 \cdot \frac{1}{3}b \cdot \frac{1}{2}b \cdot \cos(180^\circ - 2\alpha)$$

Since $\cos(180^\circ - 2\alpha) = -\cos 2\alpha = 1 - 2\cos^2 \alpha$, dividing by b^2 :

$$\frac{4}{9} = \frac{1}{9} + \frac{1}{4} - \frac{1}{3} + \frac{2}{3}\cos^2 \alpha$$

so $\cos^2 \alpha = \frac{5}{8}$, $\cos \alpha = \frac{\sqrt{5}}{2\sqrt{2}}$ (it is not negative, since α is acute). In the triangle ABC , $\cos \alpha = \frac{1/2}{b} = \frac{1}{2b}$, so $b = \frac{\sqrt{2}}{\sqrt{5}}$.

Now by the Law of Cosines for the side FX of the triangle FBX (with $FB = \frac{b}{2} = \frac{\sqrt{2}}{2\sqrt{5}}$):

$$(1-x)^2 = x^2 + \left(\frac{\sqrt{2}}{2\sqrt{5}}\right)^2 - 2x \cdot \frac{\sqrt{2}}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{2\sqrt{2}}, \quad \text{that is,} \quad 1-2x = \frac{1}{10} - \frac{x}{2}$$

so $\frac{9}{10} = \frac{3}{2}x$, and $x = \frac{3}{5}$. So $BX = \frac{3}{5}$ and $AX = \frac{2}{5}$: the other endpoint of the fold line is the point dividing AB in the ratio $2:3$, closer to A .

8. Karcsi saw a sports shoe at a very good price in a shop window. He went in to buy it, but to his great surprise the cashier wanted him to pay four times the price he had seen in the window. It turned out that the window dresser had accidentally written the digits of the four-digit price in reverse order. How much did the shoe cost?

(16 points)

Solution: If the price of the shoe is the four-digit number $\overline{abcd}$, then Karcsi saw $\overline{dcba}$ in the window, and $\overline{abcd} = 4 \cdot \overline{dcba}$. The right side is even, so d is even. Also $d \leq 2$, since otherwise the right side would be a five-digit number. So $d = 2$, and $\overline{abc2} = 4 \cdot \overline{2cba}$.

The right side is greater than 8000, so a is 8 or 9. The last digit of $4a$ must be 2, so $a = 8$: $\overline{8bc2} = 4 \cdot \overline{2cb8}$. Written out,

$$8000 + 100b + 10c + 2 = 4(2000 + 100c + 10b + 8), \quad \text{that is,} \quad 2b = 1 + 13c$$

The left side is at most 18, so $c = 1$ ($c = 0$ gives no integer b), and then $b = 7$. The shoe cost 8712 forints. (The window showed 2178, and indeed $8712 = 4 \cdot 2178$.)

9. a) The sides of the triangle ABC are 10, 12, and 15 cm long. On the sides (not counting the endpoints of the sides) we marked a point at every whole cm. Then we formed all triangles whose vertices are among the marked points. How many such triangles can we form?
- b) The sides of the equilateral triangle PQR are n cm long, where $n > 2$ is an integer. On the sides (not counting the endpoints of the sides) we marked a point at every whole cm. Then we formed all triangles whose vertices are among the marked points, and then all quadrilaterals with two vertices on one side and the other two vertices on the second and the third side, respectively. Which are there more of: triangles or quadrilaterals?

(16 points)

Solution: a) The sides have 14, 11, and 9 marked points (on the sides of length 15, 12, and 10).

If two vertices of the triangle are on the side with 14 points, they can be chosen in $\binom{14}{2}$ ways, and the third vertex is on one of the other sides, 20 choices: $20 \binom{14}{2}$ triangles. In the same way there are $23 \binom{11}{2}$ and $25 \binom{9}{2}$ triangles with two vertices on the other sides. If the three vertices are on three different sides, there are $14 \cdot 11 \cdot 9$ triangles. In all,

$$\binom{14}{2} \cdot 20 + \binom{11}{2} \cdot 23 + \binom{9}{2} \cdot 25 + 14 \cdot 11 \cdot 9 = 5371$$

- b) Each side of the equilateral triangle has $n - 1$ marked points. **Triangles:** with two vertices on one side, the side can be chosen in 3 ways, the two vertices in $\binom{n-1}{2}$ ways, and the third vertex from the other $2(n-1)$ points:

$3 \cdot 2 \binom{n-1}{2} (n-1) = 3(n-2)(n-1)^2$ triangles. With the three vertices on three different sides there are $(n-1)^3$ triangles. So there are $3(n-2)(n-1)^2 + (n-1)^3$ triangles.

Quadrilaterals: the side with two vertices can be chosen in 3 ways, the two vertices in $\binom{n-1}{2}$ ways, and the other two vertices in $(n-1)^2$ ways: $3 \binom{n-1}{2} (n-1)^2 = \frac{3(n-2)(n-1)^3}{2}$ quadrilaterals.

There are at least as many quadrilaterals as triangles if

$$\frac{3(n-2)(n-1)^3}{2} \geq 3(n-2)(n-1)^2 + (n-1)^3$$

Dividing by $(n-1)^2 > 0$ and rearranging: $3(n^2 - 3n + 2) \geq 8n - 14$, that is, $3n^2 - 17n + 20 \geq 0$, whose zeros are $\frac{5}{3}$ and 4.

So if $n = 3$, there are more triangles; if $n = 4$, there are as many triangles as quadrilaterals; and if $n > 4$, there are more quadrilaterals than triangles.