

author: János Simon

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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Show that for the function $f(x) = \frac{1}{x^2}$ (where $x \neq 0$ is a real number), for every pair of real numbers $a \neq 0$, $b \neq 0$ (with $a + b \neq 0$),

$$f(a) + f(b) + 2ab \cdot f(ab) = \frac{f(ab)}{f(a+b)}$$

(11 points)

Solution: Substituting on the left side:

$$\frac{1}{a^2} + \frac{1}{b^2} + 2ab \cdot \frac{1}{(ab)^2} = \frac{a^2 + b^2 + 2ab}{a^2b^2} = \frac{(a+b)^2}{a^2b^2} = \frac{\frac{1}{(ab)^2}}{\frac{1}{(a+b)^2}}$$

which is exactly $\frac{f(ab)}{f(a+b)}$.

2. Starting from some natural number, we choose every 12th number of the sequence of natural numbers until the sum of the chosen numbers is 2010. More than 10 but fewer than 100 numbers were chosen. Which number did we choose first, and how many numbers did we choose in all?

(12 points)

Solution: Let the first chosen number be x and the number of chosen numbers k ($10 < k < 100$). The sum of the chosen numbers is

$$x + (x + 12) + \dots + (x + (k-1)12) = kx + 12 \cdot \frac{(k-1)k}{2} = 2010$$

that is, $k(x + 6k - 6) = 2010 = 2 \cdot 3 \cdot 5 \cdot 67$. So k is a divisor of 2010 between 10 and 100: 15, 30, or 67.

If $k = 15$: $x = 134 - 84 = 50$. If $k = 30$: $x = 67 - 174 < 0$. If $k = 67$: $x = 30 - 396 < 0$.

So only $k = 15$ works: the first chosen number was 50, and we chose 15 numbers (50, 62, ..., 218).

3. Show that if the sides of a triangle satisfy $a^2 + b^2 > 5c^2$, then c is the shortest side of the triangle.

(14 points)

Solution: By the triangle inequality, $b + c > a$, so $b^2 + 2bc + c^2 > a^2$. Adding b^2 to both sides:

$$2b^2 + 2bc + c^2 > a^2 + b^2 > 5c^2$$

Rearranging, $b^2 + bc - 2c^2 > 0$, that is, $(b - c)(b + 2c) > 0$. The second factor is positive, so the first one is also positive: $b > c$. In the same way $a > c$, so c is indeed the shortest side of the triangle.

4. Does the equation $\sin^6 x + \sin^4 x + 2\sin^2 x + \cos^2 x = 4\sin^5 x$ have a solution on the interval $\left(0, \frac{\pi}{2}\right)$?

(14 points)

Solution: Dividing both sides by 4 and using $\cos^2 x = 1 - \sin^2 x$ on the left side, every term on the left is positive, and by the inequality between the arithmetic and geometric means,

$$\sin^5 x = \frac{\sin^6 x + \sin^4 x + \sin^2 x + 1}{4} \geq \sqrt[4]{\sin^{12} x} = \sin^3 x$$

that is, $\sin^3 x (\sin^2 x - 1) \geq 0$. On the given interval $0 < \sin x < 1$, so the first factor is positive and the second is negative, and the product is negative. So the equation has no solution on this interval.

Part II

5. Solve the following inequality over the real numbers.

$$x^2 \cdot \log_4 (5x^2 - 2x - 3) - x \cdot \log_{\frac{1}{4}} (5x^2 - 2x - 3) \leq x^2 + x$$

(16 points)

Solution: Because of the logarithm, $5x^2 - 2x - 3 > 0$, so $x < -0.6$ or $1 < x$. Since $\log_{\frac{1}{4}} u = -\log_4 u$, after rearranging to 0 and factoring, the inequality becomes

$$x(x+1) [\log_4 (5x^2 - 2x - 3) - 1] \leq 0$$

The base 4 logarithm is strictly increasing, so the third factor is nonnegative if $5x^2 - 2x - 3 \geq 4$, that is, $5x^2 - 2x - 7 \geq 0$, which gives $x \leq -1$ or $1.4 \leq x$. Looking at the signs of the three factors on the number line, together with the condition from the logarithm, the solution set is

$$\{-1\} \cup (1, 1.4]$$

6. Márton hits the upper hole of the “kicking wall” in the Stadium park with probability 10%, and the lower hole with probability 40%.
- Shooting once at each hole, what is the probability that he A: never hits; B: hits exactly once?
 - How many times must he shoot at the lower hole to hit at least once with probability greater than 95%?
 - What is the probability that, shooting three times at the lower hole and then three times at the upper hole, he hits exactly once?
 - Krisztián noticed that Márton’s hit probability is not constant. After a hit it increases by $\frac{1}{10}$ of the original probability, and after a miss it decreases by $\frac{1}{10}$ of the original probability. Márton shoots twice at the lower hole. What is the probability that he hits exactly once?

(16 points)

Solution: a) A: $0.9 \cdot 0.6 = 0.54$. B: $0.9 \cdot 0.4 + 0.1 \cdot 0.6 = 0.42$.

b) Let n be the number of shots. Then $1 - 0.6^n > 0.95$, that is, $0.05 > 0.6^n$, $\lg 0.05 > n \lg 0.6$, so $n > \frac{\lg 0.05}{\lg 0.6} \approx 5.86$. He must shoot at least 6 times.

c)

$$\binom{3}{1} \cdot 0.4 \cdot 0.6^2 \cdot 0.9^3 + \binom{3}{1} \cdot 0.6^3 \cdot 0.1 \cdot 0.9^2 \approx 0.367$$

d) After a hit the probability is 0.44, after a miss 0.36. So the probability of exactly one hit is $0.4 \cdot 0.56 + 0.6 \cdot 0.36 = 0.44$.

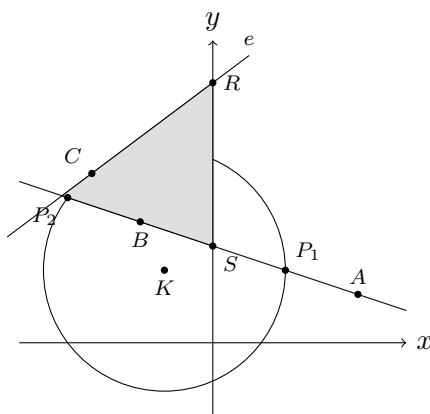
7. In a coordinate system, the circle with center $K(-2, 3)$ and radius $r = 5$ and three points are given: $A(6, 2)$, $B(-3, 5)$, and $C(-5, y_0)$, where $y_0 > 0$.

a) What is the length of the chord of the circle on the line AB ?

b) The point C lies on the circle. Let e be the tangent line of the circle at C . What is the area of the triangle enclosed by the line e , the y -axis, and the line AB ?

(16 points)

Solution: a) The line AB is $y = -\frac{1}{3}x + 4$, and the circle is $(x + 2)^2 + (y - 3)^2 = 25$. Substituting $y = -\frac{1}{3}x + 4$ into the equation of the circle: $\frac{10}{9}x^2 + \frac{10}{3}x - 20 = 0$, so $x_1 = 3$ and $x_2 = -6$. The intersection points are $P_1(3, 3)$ and $P_2(-6, 6)$, and the chord is $P_1P_2 = \sqrt{9^2 + 3^2} = 3\sqrt{10}$.

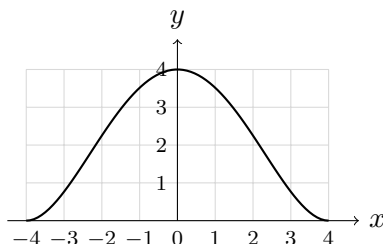


b) Since C lies on the circle, $(-5 + 2)^2 + (y_0 - 3)^2 = 25$, that is, $y_0^2 - 6y_0 - 7 = 0$, so $y_0 = 7$ or $y_0 = -1$; since $y_0 > 0$, $C(-5, 7)$. The tangent line at C is perpendicular to $\overrightarrow{KC} = (-3, 4)$: $e: 3x - 4y = -43$. It meets the line AB

$(\frac{1}{3}x + y = 4)$ at $Q(-\frac{81}{13}, \frac{79}{13})$, and the y -axis at $R(0, \frac{43}{4})$. The line AB meets the y -axis at $S(0, 4)$. So the side RS of the triangle is $\frac{27}{4}$, and the altitude to it is $\frac{81}{13}$. The area is

$$t = \frac{\frac{27}{4} \cdot \frac{81}{13}}{2} = \frac{2187}{104} \approx 21.03$$

8. The gable of a house waiting for renovation is 8 m wide and 4 m high (see the figure). The tangent line of the curve describing the arch of the gable is horizontal at the endpoints. The architect wants to describe the curve by the graph of the function $f(x) = ax^4 + bx^2 + c$.



- a) Find the constants a , b , and c in the formula of f .
 b) What is the minimum number of cans of paint needed to paint the gable twice, if 1 m^2 needs 350 cm^3 of paint, and the paint comes in 2-liter, 4-liter, and 5-liter cans?

(16 points)

Solution: a) Substituting $x = 0$: $f(0) = 4 = c$. We know that $f(-4) = f(4) = 0 = 256a + 16b + 4$. Also $f'(x) = 4ax^3 + 2bx$, and $f'(4) = 0 = 256a + 8b$. From the system

$$\begin{aligned} 256a + 16b + 4 &= 0 \\ 256a + 8b &= 0 \end{aligned}$$

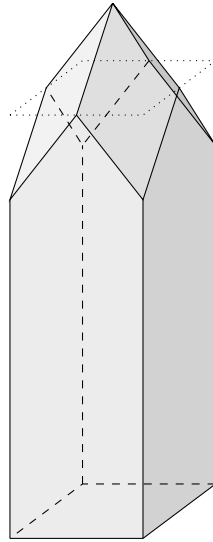
we get $a = \frac{1}{64}$, $b = -\frac{1}{2}$. So the constants are $a = \frac{1}{64}$, $b = -\frac{1}{2}$, $c = 4$.

- b) The area of the gable is

$$T = \int_{-4}^4 \left(\frac{1}{64}x^4 - \frac{1}{2}x^2 + 4 \right) dx = \left[\frac{x^5}{320} - \frac{x^3}{6} + 4x \right]_{-4}^4 = \frac{256}{15} \approx 17.07 \text{ m}^2$$

The paint needed for two coats is $17.07 \cdot 2 \cdot 350 \approx 11947 \text{ cm}^3$, about 11.95 liters. So at least 3 cans are needed (two 5-liter cans and one 2-liter can, or three 4-liter cans).

9. Prince Architectos and his three sons captured a city with their armies. At three corners of the city wall stood three identical towers in the shape of square prisms; the edges of their bases were 8 m and their heights 40 m. To remember the victory, the three sons rebuilt the three towers, in very similar ways. Along the segments joining the midpoints of neighboring edges of the top face of each original tower, they cut off four identical corners. These were placed on the remaining top face as in the figure, so that the new roof of the tower consisted of rhombi joined to each other.



The surface area of the tower of the oldest son did not change after the rebuilding. The middle son rebuilt his tower so that its total surface area became as small as possible. On the tower of the youngest son, neighboring roof planes made an angle of 120° . Find the heights of the rebuilt towers.

(16 points)

Solution: Let h be how much higher the new tower is than the original one. Each rhombus has one diagonal $4\sqrt{2}$ (joining two neighboring edge midpoints) and the other $2b$, where b is half of the other diagonal. The surface area of the original tower is $A = 4 \cdot 40 \cdot 8 + 8 \cdot 8 = 1344$. The area of the new roof is

$$4 \cdot \frac{4\sqrt{2} \cdot 2b}{2} = 16\sqrt{2}b$$

and by the Pythagorean theorem $b = \sqrt{8 + h^2}$. The original surface loses the top face and 8 right triangles with legs 4 and h on the lateral faces:

$$8 \cdot 8 + 8 \cdot \frac{4h}{2} = 64 + 16h$$

Oldest son. The surface area does not change if $64 + 16h = 16\sqrt{2} \cdot \sqrt{8 + h^2}$, that is, $4 + h = \sqrt{16 + 2h^2}$. Squaring, $16 + 8h + h^2 = 16 + 2h^2$, that is, $h(h - 8) = 0$. Only $h = 8$ is possible, so the oldest son's tower became 48 m high.

Middle son. The surface area of the new tower is $A(h) = 1344 - (64 + 16h) + 16\sqrt{16 + 2h^2} = 1280 - 16h + 16\sqrt{16 + 2h^2}$. Its derivative is

$$A'(h) = -16 + \frac{32h}{\sqrt{16 + 2h^2}} = \frac{16}{\sqrt{16 + 2h^2}} (2h - \sqrt{16 + 2h^2})$$

which is 0 if $2h = \sqrt{16 + 2h^2}$, that is, $h^2 = 8$, $h = 2\sqrt{2}$. Here the derivative changes from negative to positive, so the surface area is indeed minimal. The middle son's tower became $40 + 2\sqrt{2} \approx 42.8$ m high.

Youngest son. The side of a rhombus is $\sqrt{16 + h^2}$, and its altitude is m ; from the area of the rhombus,

$$\sqrt{16 + h^2} \cdot m = \frac{4\sqrt{2} \cdot 2b}{2} = 4\sqrt{2} \cdot \sqrt{8 + h^2}, \quad \text{so} \quad m = \frac{4\sqrt{16 + 2h^2}}{\sqrt{16 + h^2}}$$

The altitudes of two neighboring rhombi, drawn to their common side from a common foot, make the angle 120° ; they form an isosceles triangle with legs m and base 8 (the diagonal of the half top face). So

$$\sin 60^\circ = \frac{4}{m} = \sqrt{\frac{16 + h^2}{16 + 2h^2}} = \frac{\sqrt{3}}{2}, \quad \text{that is,} \quad \frac{16 + h^2}{16 + 2h^2} = \frac{3}{4}$$

so $64 + 4h^2 = 48 + 6h^2$, $h^2 = 8$, and $h = 2\sqrt{2}$. The youngest son's tower also became $40 + 2\sqrt{2} \approx 42.8$ m high.