

author: Ágnes Benyovics

translated from: KoMaL, April 2010, pages 208-209

Pdf file: KoMaL, April 2010, pages 208-209

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The edge of a cube, measured in centimeters, is an integer. We painted the surface of the cube red, and then cut it into small cubes of edge 1 cm with cuts parallel to its faces.
 - a) What was the edge of the cube, if the number of small cubes with exactly one painted face equals the number of unpainted small cubes?
 - b) How many cubes of different sizes are there from which we can choose an unpainted small cube with probability between 0.5 and 0.6? (Each small cube is chosen with equal probability.)

(11 points)

2. Prove that for every natural number n
 - a) the number $n^5 - 5n^3 + 4n + 1$ ends in 1;
 - b) $5 \mid 2^{4n+1} + 3$.

(12 points)

3. Solve the following inequality.

$$\log_{x+3} (x^2 - 9x - 10) \leq \log_{x+3} 12$$

(14 points)

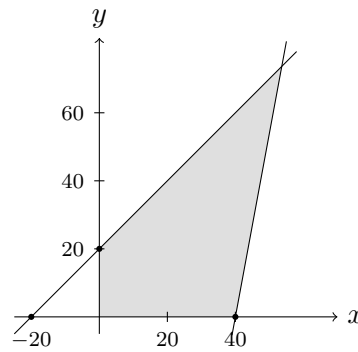
4.
 - a) The first, fourth, and tenth terms of an increasing arithmetic sequence are the first three terms of a geometric sequence. The eighth term of the arithmetic sequence is 10. Find the first term of the geometric sequence.
 - b) From the digits 3, 4, 5, 6 we randomly form a four-digit number with all different digits. What is the probability that the number is divisible by 4?

(14 points)

Part II

5. On a corner lot in the shape of a right triangle we design a house with a rectangular base, so that two of its sides are parallel to the streets. The two perpendicular sides of the lot are 50 m and 30 m long. How should we choose the side lengths of the house to get the largest possible base area?
6. The two lines in the figure cut off a quadrilateral of area 2010 units from the first quadrant of the coordinate system.

(16 points)



- a) Find the coordinates of the intersection point of the two lines.
- b) Write the equations of the lines.

(16 points)

7. We choose two numbers at random in the interval $[0, 1]$. What is the probability that their sum is less than 1 and their product is less than $\frac{4}{25}$?

(16 points)

8. A right circular cone and a cylinder have a common base. The apex of the cone is the center of the top face of the cylinder. Find the angle between the axis and a generator of the cone, if the surface areas of the cylinder and the cone are in the ratio 7 : 4.

(16 points)

9. We rotate the graphs of the functions $f(x) = x^3 + 1$ and $g(x) = (x + 1)^2$, defined on the interval $[0, 2]$, about the x -axis.

- a) Find the areas of the base and top faces of the resulting solids of revolution.
- b) Find the volumes of the two solids.

(16 points)