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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The edge of a cube, measured in centimeters, is an integer. We painted the surface of the cube red, and then cut it into small cubes of edge 1 cm with cuts parallel to its faces.
 - a) What was the edge of the cube, if the number of small cubes with exactly one painted face equals the number of unpainted small cubes?
 - b) How many cubes of different sizes are there from which we can choose an unpainted small cube with probability between 0.5 and 0.6? (Each small cube is chosen with equal probability.)

(11 points)

Solution: a) Let the edge of the cube be x cm ($x \in \mathbb{Z}^+$). The number of unpainted small cubes is $(x-2)^3$, and the number of small cubes with exactly one painted face is $6(x-2)^2$. So $(x-2)^3 = 6(x-2)^2$, that is, $(x-2)^2(x-8) = 0$. $x = 2$ is not a solution of the problem, so the only solution is $x = 8$: the edge of the cube was 8 cm.

b) Let the probability be p , with $0.5 \leq p \leq 0.6$:

$$0.5 \leq \frac{(x-2)^3}{x^3} \leq 0.6, \quad \text{that is,} \quad \sqrt[3]{0.5} \cdot x + 2 \geq x \geq \sqrt[3]{0.6} \cdot x + 2$$

which gives $9.69 \leq x \leq 12.77$ (to two decimal places). So x can be 10, 11, or 12: there are 3 such cubes.

2. Prove that for every natural number n
 - a) the number $n^5 - 5n^3 + 4n + 1$ ends in 1;
 - b) $5 \mid 2^{4n+1} + 3$.

(12 points)

Solution: a) $n^5 - 5n^3 + 4n + 1$ ends in 1 exactly when $n^5 - 5n^3 + 4n$ ends in 0, that is, when it is divisible by 10:

$$n^5 - 5n^3 + 4n = n(n^4 - 5n^2 + 4) = n(n^2 - 1)(n^2 - 4) = (n-2)(n-1)n(n+1)(n+2)$$

This is the product of five consecutive integers, so one of the factors is divisible by 2 and one by 5; so the expression is divisible by 10 for every natural number n .

b) The powers of 2 end in 2, 4, 8, 6, 2, 4, ... in turn, so 2^{4n+1} always ends in 2, and $2^{4n+1} + 3$ always ends in 5. So the expression is indeed divisible by 5 for every natural number n .

3. Solve the following inequality.

$$\log_{x+3} (x^2 - 9x - 10) \leq \log_{x+3} 12$$

(14 points)

Solution: The domain: $x^2 - 9x - 10 = (x + 1)(x - 10) > 0$, that is, $x < -1$ or $x > 10$; $x + 3 > 0$, that is, $x > -3$; and $x + 3 \neq 1$, that is, $x \neq -2$. So $x \in (-3, -2) \cup (-2, -1) \cup (10, \infty)$.

If $x \in (-3, -2)$, the base is less than 1, so the logarithm function is decreasing: $x^2 - 9x - 10 \geq 12$, that is, $(x + 2)(x - 11) \geq 0$, so $x \leq -2$ or $x \geq 11$. In this case the solution is $x \in (-3, -2)$.

If $x \in (-2, -1)$, the logarithm function is increasing: $x^2 - 9x - 10 \leq 12$, that is, $(x + 2)(x - 11) \leq 0$, so $-2 \leq x \leq 11$. In this case the solution is $x \in (-2, -1)$.

If $x \in (10, \infty)$, the logarithm function is increasing: $x^2 - 9x - 10 \leq 12$, so $-2 \leq x \leq 11$. In this case the solution is $x \in (10, 11]$.

Altogether, the solution is $x \in (-3, -2) \cup (-2, -1) \cup (10, 11]$.

4. a) The first, fourth, and tenth terms of an increasing arithmetic sequence are the first three terms of a geometric sequence. The eighth term of the arithmetic sequence is 10. Find the first term of the geometric sequence.
- b) From the digits 3, 4, 5, 6 we randomly form a four-digit number with all different digits. What is the probability that the number is divisible by 4?

(14 points)

Solution: a) Let the terms of the arithmetic sequence be $a_1, a_2, \dots$ with difference d . Then $a_4 = a_1 + 3d$, $a_{10} = a_1 + 9d$, and $a_8 = a_1 + 7d = 10$, so $a_1 = 10 - 7d$. By the property of the geometric sequence,

$$\frac{a_1 + 3d}{a_1} = \frac{a_1 + 9d}{a_1 + 3d}, \quad \text{that is,} \quad \frac{10 - 4d}{10 - 7d} = \frac{10 + 2d}{10 - 4d}$$

which gives $30d^2 - 30d = 0$, that is, $30d(d - 1) = 0$. $d = 0$ is not a solution, since the sequence is increasing; so $d = 1$, and the first term of the geometric sequence is $a_1 = 10 - 7d = 3$. (The geometric sequence is 3, 6, 12.)

b) There are $4! = 24$ four-digit numbers with different digits from 3, 4, 5, 6. A number is divisible by 4 if the number formed by its last two digits is; the possible endings are 36, 56, and 64. In each case the other two digits can be written in front in 2 ways, so there are 6 favorable cases, and

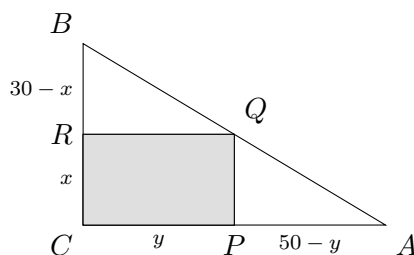
$$p = \frac{6}{24} = 0.25$$

Part II

5. On a corner lot in the shape of a right triangle we design a house with a rectangular base, so that two of its sides are parallel to the streets. The two perpendicular sides of the lot are 50 m and 30 m long. How should we choose the side lengths of the house to get the largest possible base area?

(16 points)

Solution: Let the lot be the right triangle ABC with $CA = 50$, $CB = 30$, and let the house be the rectangle $CPQR$ with $CR = x$, $CP = y$, and Q on the hypotenuse.



The triangles APQ and QRB are similar, so

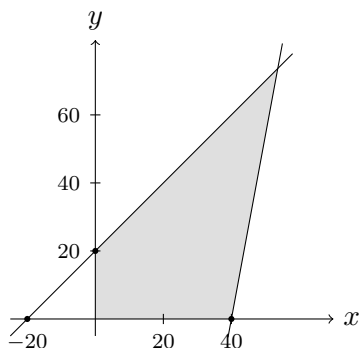
$$\frac{30 - x}{x} = \frac{y}{50 - y}, \quad \text{that is,} \quad y = \frac{150 - 5x}{3}$$

We look for the largest area:

$$t(x) = xy = \frac{x(150 - 5x)}{3} = -\frac{5}{3}x^2 + 50x$$

The zeros of this quadratic function are 0 and 30, so by symmetry its extremum is at $x = 15$, and it is a maximum since the leading coefficient is negative. The sides of the house are $x = 15$ m and $y = 25$ m (with area 375 m^2).

6. The two lines in the figure cut off a quadrilateral of area 2010 units from the first quadrant of the coordinate system.

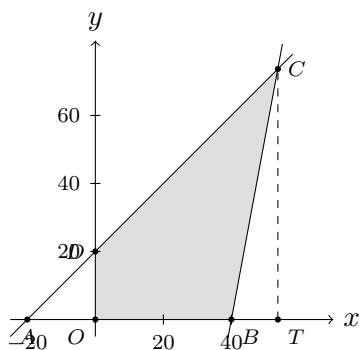


a) Find the coordinates of the intersection point of the two lines.

b) Write the equations of the lines.

(16 points)

Solution: Let $A(-20, 0)$, $D(0, 20)$, $B(40, 0)$, and let $C(c_1, c_2)$ be the intersection point of the lines.



$$t_{ABC} = \frac{60 \cdot c_2}{2} = 30c_2, \quad t_{AOD} = \frac{20 \cdot 20}{2} = 200, \quad t_{OBCD} = t_{ABC} - t_{AOD} = 30c_2 - 200 = 2010$$

so $c_2 = \frac{221}{3}$. The triangle ATC is an isosceles right triangle (the line AD has slope 1), so $AT = CT$, that is, $20 + c_1 = \frac{221}{3}$, and $c_1 = \frac{161}{3}$.

a) The intersection point of the two lines is $C\left(\frac{161}{3}, \frac{221}{3}\right)$.

b) The line AC has slope 1 and meets the y -axis at $D(0, 20)$: $y = x + 20$. The line BC passes through $B(40, 0)$ and C :

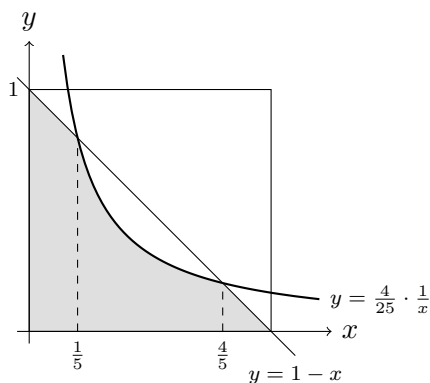
$$y\left(\frac{161}{3} - 40\right) = (x - 40)\frac{221}{3}, \quad \text{that is,} \quad 221x - 41y = 8840$$

So the equations of the lines are $y = x + 20$ and $221x - 41y = 8840$.

7. We choose two numbers at random in the interval $[0, 1]$. What is the probability that their sum is less than 1 and their product is less than $\frac{4}{25}$?

(16 points)

Solution: Let A be the event in question. We represent the two random numbers x and y by the point (x, y) of the unit square. A occurs if $x + y < 1$ and $xy < \frac{4}{25}$ both hold; these points form the shaded part of the unit square, and we need its area.



The line $x + y = 1$ and the hyperbola $xy = \frac{4}{25}$ meet where $x(1 - x) = \frac{4}{25}$, that is, $x^2 - x + \frac{4}{25} = 0$, so $x = 0.8$ or $x = 0.2$. The parts of the shaded region above the intervals $[0, 0.2]$ and $[0.8, 1]$ (under the line) together have area 0.2. On the interval $[0.2, 0.8]$ we need the area under the hyperbola $y = 0.16 \cdot \frac{1}{x}$:

$$0.16 \int_{0.2}^{0.8} \frac{1}{x} dx = 0.16 [\ln |x|]_{0.2}^{0.8} = 0.16 (\ln 0.8 - \ln 0.2) = 0.16 \ln 4$$

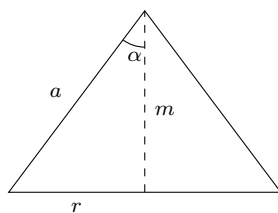
The area of the favorable region is $t = 0.2 + 0.16 \ln 4$, and the area of the unit square is 1, so

$$P(A) = 0.2 + 0.16 \ln 4 \approx 0.42$$

8. A right circular cone and a cylinder have a common base. The apex of the cone is the center of the top face of the cylinder. Find the angle between the axis and a generator of the cone, if the surface areas of the cylinder and the cone are in the ratio 7 : 4.

(16 points)

Solution: Consider the cross section of the cone through its axis; let r be the radius, m the height, a the slant height, and α the angle between the axis and a generator.



By the formulas for the surface areas,

$$\frac{A_{\text{cylinder}}}{A_{\text{cone}}} = \frac{2r^2\pi + 2r\pi m}{r^2\pi + r\pi a} = \frac{7}{4}, \quad \text{that is,} \quad \frac{2(r + m)}{r + a} = \frac{7}{4}, \quad \frac{1 + \frac{m}{r}}{1 + \frac{a}{r}} = \frac{7}{8}$$

Since $\cot \alpha = \frac{m}{r}$ and $\frac{1}{\sin \alpha} = \frac{a}{r}$:

$$\frac{1 + \frac{\cos \alpha}{\sin \alpha}}{1 + \frac{1}{\sin \alpha}} = \frac{7}{8}, \quad \text{that is,} \quad \sin \alpha + 8 \cos \alpha = 7$$

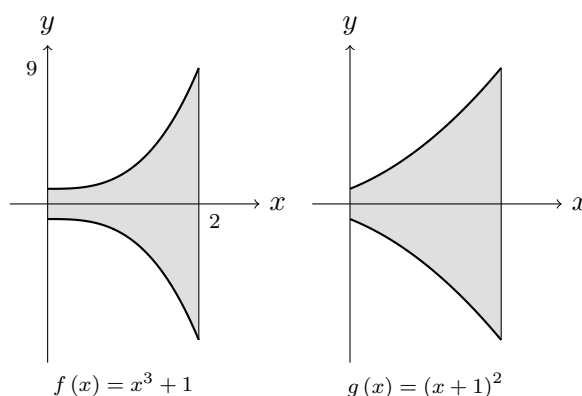
Dividing by $\sqrt{1^2 + 8^2} = \sqrt{65}$ and using approximate values, $\sin \alpha \cos 82.87^\circ + \cos \alpha \sin 82.87^\circ = 0.8682$, that is, $\sin(82.87^\circ + \alpha) = \sin 60.25^\circ$. The solutions of this equation are $\alpha = -22.62^\circ + k_1 \cdot 360^\circ$ and $\alpha = 36.87^\circ + k_2 \cdot 360^\circ$, so the only solution of the problem is $\alpha \approx 36.87^\circ$. (Exactly: $\sin \alpha = \frac{3}{5}$, $\cos \alpha = \frac{4}{5}$.)

9. We rotate the graphs of the functions $f(x) = x^3 + 1$ and $g(x) = (x+1)^2$, defined on the interval $[0, 2]$, about the x -axis.

- Find the areas of the base and top faces of the resulting solids of revolution.
- Find the volumes of the two solids.

(16 points)

Solution: a) The base and top faces of the two solids are the same size, since $f(0) = g(0) = 1$ and $f(2) = g(2) = 9$. So the radii of the base faces are 1 and those of the top faces 9; the area of the base face is π , and the area of the top face is 81π .



- b) Using $V = \pi \int_a^b [f(x)]^2 dx$:

$$V_1 = \pi \int_0^2 (x^6 + 2x^3 + 1) dx = \pi \left[\frac{x^7}{7} + \frac{x^4}{2} + x \right]_0^2 = \pi \left(\frac{128}{7} + 8 + 2 \right) = \frac{198}{7}\pi \approx 88.86$$

$$V_2 = \pi \int_0^2 (x+1)^4 dx = \pi \left[\frac{(x+1)^5}{5} \right]_0^2 = \frac{242}{5}\pi \approx 152.05$$