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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the exact values of the following expressions without a calculator.

$$\text{a) } \sqrt{100^{\lg(1-\sin \frac{19\pi}{6})}} \quad \text{b) } \cos^2 15^\circ \cdot \tan 15^\circ \quad \text{c) } \log_\pi \frac{1}{\sqrt{\pi}} + \tan\left(-\frac{11\pi}{4}\right) + 2^{\log_4 9}$$

(11 points)

Solution: a) $\sin \frac{19\pi}{6} = \sin\left(\frac{7\pi}{6} + 2\pi\right) = \sin \frac{7\pi}{6} = -\frac{1}{2}$, so the exponent is $\lg \frac{3}{2}$, and

$$\sqrt{100^{\lg \frac{3}{2}}} = 10^{\lg \frac{3}{2}} = \frac{3}{2}$$

b) Using $\tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ}$ and $2 \sin 15^\circ \cos 15^\circ = \sin 30^\circ$:

$$\cos^2 15^\circ \cdot \frac{\sin 15^\circ}{\cos 15^\circ} = \cos 15^\circ \sin 15^\circ = \frac{\sin 30^\circ}{2} = \frac{1}{4}$$

c) $\log_\pi \frac{1}{\sqrt{\pi}} = \log_\pi \pi^{-\frac{1}{2}} = -\frac{1}{2}$, $\tan\left(-\frac{11\pi}{4}\right) = \tan\left(\frac{\pi}{4} - 3\pi\right) = \tan \frac{\pi}{4} = 1$, and $2^{\log_4 9} = 4^{\log_4 9^{\frac{1}{2}}} = 9^{\frac{1}{2}} = 3$. So the value is $-\frac{1}{2} + 1 + 3 = \frac{7}{2}$.

2. Solve the following system of equations over the pairs of real numbers.

$$\begin{aligned} \sqrt{x^2 - 3xy + 4} + x(x - 3y) &= 2 \\ 2x - y &= 3 \end{aligned}$$

(12 points)

Solution: The first equation can be written as $\sqrt{x^2 - 3xy + 4} + (x^2 - 3xy + 4) - 6 = 0$. With the new unknown $a = \sqrt{x^2 - 3xy + 4} \geq 0$, this is $a^2 + a - 6 = 0$, with solutions $a = -3$ and $a = 2$; a cannot be negative. So $\sqrt{x^2 - 3xy + 4} = 2$, that is, $x^2 - 3xy = 0$.

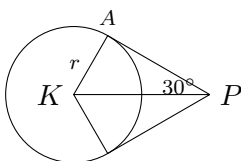
From the second equation $y = 2x - 3$; substituting, $x^2 - 3x(2x - 3) = 0$, whose solutions are $x_1 = 0$ and $x_2 = \frac{9}{5}$. So the solutions of the system are

$$(0, -3) \quad \text{and} \quad \left(\frac{9}{5}, \frac{3}{5}\right)$$

3. The circle $x^2 + y^2 - 6x - 4y + a = 0$ is given. There is exactly one point on the x -axis from which the tangent lines drawn to the circle make an angle of 60° with each other.
- a) Find the value of the parameter a .
- b) For $a = 12$, what angle do the tangent lines drawn to the circle from the origin make with each other?

(14 points)

Solution: a) The circle is $(x - 3)^2 + (y - 2)^2 = 13 - a$, with center $K(3, 2)$ and radius $r = \sqrt{13 - a}$. If the tangent lines from a point P make 60° with each other, then $\sin 30^\circ = \frac{1}{2} = \frac{r}{KP}$, so $KP = 2r$.



So the possible points P are on the circle with center K and radius $2r$: $(x - 3)^2 + (y - 2)^2 = 4(13 - a)$. This circle has exactly one common point with the x -axis if the second coordinate of its center equals its radius: $2 = 2\sqrt{13 - a}$, so $a = 12$.

- b) For $a = 12$ the circle is $(x - 3)^2 + (y - 2)^2 = 1$. Let the angle of the tangent lines be 2α ; then

$$\sin \alpha = \frac{1}{OK} = \frac{1}{\sqrt{3^2 + 2^2}} = \frac{1}{\sqrt{13}}, \quad \text{so} \quad \alpha \approx 16.1^\circ$$

The tangent lines from the origin make an angle of about 32.2° .

4. Three positive numbers are three consecutive terms of a geometric sequence. If we add two to the second one, we get three consecutive terms of an arithmetic sequence. If we then increase the first term of this new sequence by 16, we again get three consecutive terms of a new geometric sequence. What are the original three numbers?

(14 points)

Solution: Let the terms of the arithmetic sequence be $a - d$, a , $a + d$. The terms of the first geometric sequence are $a - d$, $a - 2$, $a + d$, so $(a - 2)^2 = (a - d)(a + d)$. The terms of the second geometric sequence are $a - d + 16$, a , $a + d$, so $a^2 = (a + d)(a - d + 16)$.

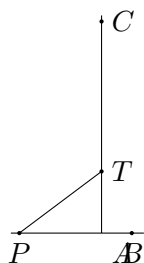
The first equation is $d^2 + 4 = 4a$, and the second $d^2 - 16a - 16d = 0$. Eliminating a : $3d^2 + 16d + 16 = 0$, so $d_1 = -4$ and $d_2 = -\frac{4}{3}$. With $d = -\frac{4}{3}$ the original sequence would have a negative term, so it does not satisfy the conditions. For $d = -4$, $a = 5$, and the three original numbers are 9, 3, 1.

Part II

5. Two straight roads cross each other at a right angle at the point A . There is a restaurant on one road 70 km from the crossing, and on the other road 10 km from the crossing. A group starts from each restaurant toward A at the same time. The group from the farther restaurant covers 4 km per hour, and the group from the nearer one 3 km per hour.
- a) After how many hours will they be closest to each other, and what is the smallest distance between them?
- b) How many hours after the start will the distance between them be 50 km?

(16 points)

Solution: a) Let C and B be the two restaurants, and let x be the number of hours after the start. The groups are at T and P , with $TA = |70 - 4x|$ and $AP = |10 - 3x|$.



By the Pythagorean theorem,

$$TP = \sqrt{(70 - 4x)^2 + (10 - 3x)^2} = \sqrt{25x^2 - 620x + 5000} = \sqrt{(5x - 62)^2 + 1156}$$

This is smallest at $x = \frac{62}{5} = 12.4$: they are closest 12.4 hours after the start, and then the distance is $\sqrt{1156} = 34$ km. (By then the second group has passed the crossing.)

b) $(70 - 4x)^2 + (10 - 3x)^2 = 2500$ has two solutions, $x_1 \approx 5.07$ and $x_2 \approx 19.73$ hours; both satisfy the conditions.

6. We roll a die three times.

- What is the probability that the product of the numbers rolled is a prime?
- What is the probability that the product of the numbers rolled is a perfect square?

(16 points)

Solution: There are $6^3 = 216$ outcomes.

- The product of the three numbers is a prime only if two of them are 1 and one is a prime: 1, 1, 2 or 1, 1, 3 or 1, 1, 5. Each case can occur in 3 ways, so there are 9 favorable outcomes, and $p = \frac{9}{216} \approx 0.0417$.
- The product is a perfect square for the following numbers: 1, 1, 1; 4, 4, 4; 1, 1, 4; 1, 2, 2; 1, 3, 3; 1, 4, 4; 1, 5, 5; 1, 6, 6; 2, 2, 4; 3, 3, 4; 5, 5, 4; 6, 6, 4; and 2, 3, 6. The first two can occur in only one way, the last one in six ways, and the others in three ways each. So there are $1 + 1 + 6 + 3 \cdot 10 = 38$ favorable outcomes, and

$$P = \frac{38}{216} = \frac{19}{108} \approx 0.176$$

7. We pour water into a container in the shape of a right circular cylinder of radius 10 cm. Then we place into it a right square pyramid all of whose edges are 12 cm. The pyramid sinks to the bottom of the water, its base square stands on the base circle of the cylinder, and the apex of the pyramid sticks out 3 cm above the water. How much does the water level rise?

(16 points)

Solution: The water level rises by the height of a cylinder of radius 10 cm whose volume equals the volume of the part of the pyramid in the water, a frustum. The half diagonal of the base is $\frac{12\sqrt{2}}{2} = 6\sqrt{2}$, so the height of the pyramid is $M = \sqrt{12^2 - (6\sqrt{2})^2} = 6\sqrt{2}$. The part above the water is a pyramid of height 3, similar to the whole pyramid with ratio $3 : 6\sqrt{2}$, so its base edge is $x = \frac{12 \cdot 3}{6\sqrt{2}} = 3\sqrt{2}$.

The volume of the whole pyramid is $\frac{144 \cdot 6\sqrt{2}}{3} = 288\sqrt{2}$, and that of the part above the water is $\frac{(3\sqrt{2})^2 \cdot 3}{3} = 18$. So the volume of the frustum is $288\sqrt{2} - 18 \approx 389.29$ cm³. From $10^2\pi h = 389.29$, $h \approx 1.24$ cm: the water level rises by about 1.24 cm.

Note: With edges of 12 cm the diagonal of the base square is $12\sqrt{2} \approx 16.97$ cm, less than the 20 cm diameter of the cylinder, so the vertices of the base cannot actually lie on the base circle; this condition is not needed for the solution.

8. In the equation $\sin^2 x + \sin x + 2^m - 4 = 0$, find the value of the parameter m so that the equation has exactly one root in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(16 points)

Solution: The equation has a real root if its discriminant is nonnegative: $17 - 4 \cdot 2^m \geq 0$, that is, $m \leq \log_2 \frac{17}{4} \approx 2.087$. The roots of the quadratic equation in $\sin x$ are

$$(1) \quad \sin x = \frac{-1 + \sqrt{17 - 4 \cdot 2^m}}{2}, \quad (2) \quad \sin x = \frac{-1 - \sqrt{17 - 4 \cdot 2^m}}{2}$$

The sine function is increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and takes every value of $[-1, 1]$ exactly once there, so we count the roots in $[-1, 1]$.

Root (1) is in $[-1, 1]$ if $\sqrt{17 - 4 \cdot 2^m} \leq 3$, that is, $m \geq 1$. Root (2) is in $[-1, 1]$ if $\sqrt{17 - 4 \cdot 2^m} \leq 1$, that is, $m \geq 2$. So for $1 \leq m < 2$ exactly one root is in the interval; for $2 \leq m < \log_2 \frac{17}{4}$ there are two; and for $m = \log_2 \frac{17}{4}$ the two roots coincide ($\sin x = -\frac{1}{2}$).

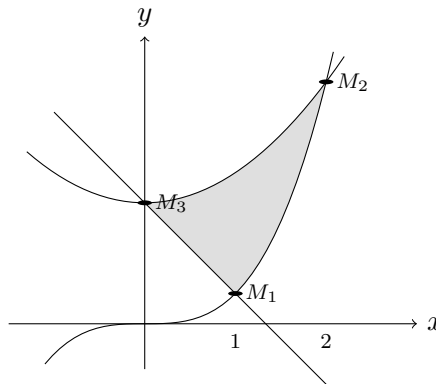
So the equation has exactly one root in the interval if $1 \leq m < 2$ or $m = \log_2 \frac{17}{4} \approx 2.09$.

Note: KoMaL gives $\log_2 \frac{17}{4} \approx 2.04$; its value is about 2.087.

9. Find the area of the finite region bounded by the graphs of the curves $y = x^2 + 4$, $y = -3x + 4$, and $y = x^3$. We rotate this region about the x -axis. What is the volume of the resulting solid of revolution?

(16 points)

Solution: The curves meet at $M_1(1, 1)$, $M_2(2, 8)$, and $M_3(0, 4)$.



$$T = \int_0^2 (x^2 + 4) dx - \int_0^1 (-3x + 4) dx - \int_1^2 x^3 dx = \left[\frac{x^3}{3} + 4x \right]_0^2 - \left[-\frac{3x^2}{2} + 4x \right]_0^1 - \left[\frac{x^4}{4} \right]_1^2 = \frac{53}{12}$$

$$\begin{aligned} V &= \pi \left(\int_0^2 (x^2 + 4)^2 dx - \int_0^1 (-3x + 4)^2 dx - \int_1^2 x^6 dx \right) \\ &= \pi \left(\left[\frac{x^5}{5} + \frac{8x^3}{3} + 16x \right]_0^2 - [3x^3 - 12x^2 + 16x]_0^1 - \left[\frac{x^7}{7} \right]_1^2 \right) = \frac{3632}{105} \pi \approx 108.67 \end{aligned}$$