

author: László Számadó

translated from: KoMaL, January 2010, pages 4-5

Pdf file: KoMaL, January 2010, pages 4-5

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. a) Prove that $\sqrt[4]{8}$, $\sqrt[4]{18}$, and $\sqrt[4]{50}$ can be the side lengths of a right triangle.
b) Prove that the following two numbers are equal.

$$\frac{\sqrt{6} - \sqrt{2}}{4} \quad \text{and} \quad \frac{2\sqrt{3} - 3}{2\sqrt{6} - 3\sqrt{3}}$$

(10 points)

2. a) Solve over the integers: $(5x + 60)^2 = (-x - 192)^2$.
b) Solve over the real numbers: $\sqrt{5x + 60} = \sqrt{-x - 192}$.
c) Solve the equation $\sin(5\alpha + 60^\circ) = \sin(-\alpha - 192^\circ)$, if $\alpha \in [-180^\circ, 180^\circ]$.

(13 points)

3. In a class of 32 students, the average height of the girls is 165 cm, and that of the boys is 172 cm.
a) Give the narrowest interval in which the average height of the class can fall.
b) During the year a 170 cm tall girl joined the class. The average height of the girls remained an integer. How many girls could there have been in the class originally?

(14 points)

4. The edges of the rectangular base of a right prism are 3 and 5, and its space diagonal is $\frac{\sqrt{497}}{2}$.
a) What angle does the space diagonal make with the shorter base edge?
b) What is the surface area of the solid?
c) We want to design a square prism with the same height and the same volume as this prism. By what percent must the base edges be changed?

(14 points)

Part II

5. The functions $f(x) = x^3$, defined on the nonnegative real numbers, and $g(x) = mx - 2m + 8$, defined on the real numbers, are given (m is a real parameter). The area of the region bounded by the graphs of the functions and the x -axis is 2010. Find the value of the parameter m .

(16 points)

6. From one corner of a rectangular sheet of paper of size 5 cm by 12 cm we measure 3 cm on the shorter side and 4 cm on the longer side. Along the line through the two points obtained in this way we would like to fold this corner of the sheet onto the diagonal opposite to the corner. Can this be done? Where does the corner go?

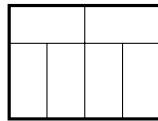
(16 points)

7. The isosceles triangles $AB_1C_1, AB_2C_2, \dots, AB_nC_n, \dots$ (n a positive integer) are given. For every n , the base AB_n of the triangle lies on the positive half of the x -axis so that the vertex A is at the origin, and the base has length $2n$. We also know that the third vertex is on the graph of the function $f(x) = x^2 + 5$.

- a) Find the perimeter and the area of the triangle AB_1C_1 .
- b) Starting from which triangle is the perimeter greater than 190?
- c) Prove that the area of every triangle is divisible by six.

(16 points)

8. The figure shows the shape of the windows of a school.



One class decided to put at most one glass sticker in the middle of each rectangular glass pane of the windows.

- a) In how many ways can the stickers be placed on a window if they plan four stickers on the six panes and the stickers are identical?
- b) In how many ways can the stickers be placed on a window if they plan three stickers on the six panes and the stickers are different?
- c) In how many ways can the stickers be placed if the panes with (identical) stickers are chosen symmetrically about the vertical axis of the window?
- d) All six parts of the window can be opened separately. They choose two at random (with equal probability) to open for ventilation. What is the probability that, for a window with three stickers, they open two undecorated parts?

(16 points)

9. The horizontal part of Main Square, in the shape of an equilateral triangle, is covered with decorative paving. In the middle of the triangle stands a tall flagpole. At each vertex of the triangle there is a person about 180 cm tall. Two of them start walking toward the flagpole. One stops when he sees the top of the pole at an angle of elevation of 72° . The other stops when he sees the top of the pole at 65° . Their companion who stayed in place sees the top of the flagpole at 50° . Then the area of the triangle determined by the three people is 23.74 m^2 .

- a) How tall is the flagpole?
- b) What is the area of the decorated part?

(16 points)