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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. a) Prove that $\sqrt[4]{8}$, $\sqrt[4]{18}$, and $\sqrt[4]{50}$ can be the side lengths of a right triangle.
- b) Prove that the following two numbers are equal.

$$\frac{\sqrt{6} - \sqrt{2}}{4} \quad \text{and} \quad \frac{2\sqrt{3} - 3}{2\sqrt{6} - 3\sqrt{3}}$$

(10 points)

Solution: a) Let $a = \sqrt[4]{8}$, $b = \sqrt[4]{18}$, and $c = \sqrt[4]{50}$. Then $c^2 = \sqrt{50}$ and

$$a^2 + b^2 = \sqrt{8} + \sqrt{18} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2} = \sqrt{50} = c^2$$

so by the converse of the Pythagorean theorem, $\sqrt[4]{8}$, $\sqrt[4]{18}$, and $\sqrt[4]{50}$ can be the side lengths of a right triangle.

b) We multiply both numbers by $4\sqrt{6 - 3\sqrt{3}}$:

$$\begin{aligned} \frac{\sqrt{6} - \sqrt{2}}{4} \cdot 4\sqrt{6 - 3\sqrt{3}} &= \sqrt{(\sqrt{6} - \sqrt{2})^2 (6 - 3\sqrt{3})} = \sqrt{(8 - 4\sqrt{3})(6 - 3\sqrt{3})} \\ &= 2\sqrt{21 - 12\sqrt{3}} = 2\sqrt{(2\sqrt{3} - 3)^2} = 2(2\sqrt{3} - 3) \\ \frac{2\sqrt{3} - 3}{2\sqrt{6} - 3\sqrt{3}} \cdot 4\sqrt{6 - 3\sqrt{3}} &= 2(2\sqrt{3} - 3) \end{aligned}$$

So the two original numbers are equal.

2. a) Solve over the integers: $(5x + 60)^2 = (-x - 192)^2$.
- b) Solve over the real numbers: $\sqrt{5x + 60} = \sqrt{-x - 192}$.
- c) Solve the equation $\sin(5\alpha + 60^\circ) = \sin(-\alpha - 192^\circ)$, if $\alpha \in [-180^\circ, 180^\circ]$.

(13 points)

Solution: a) There are two cases: $5x + 60 = -x - 192$ or $5x + 60 = -(-x - 192)$. So the solutions are $x_1 = -42$ and $x_2 = 33$.

b) Because of the square roots, $5x + 60 \geq 0$ and $-x - 192 \geq 0$, that is, $x \geq -12$ and $x \leq -192$. There is no such real number, so the equation has no real solution.

c) There are two cases.

I. $(5\alpha + 60^\circ) - (-\alpha - 192^\circ) = k_1 \cdot 360^\circ$: $6\alpha = -252^\circ + k_1 \cdot 360^\circ$, so $\alpha = -42^\circ + k_1 \cdot 60^\circ$.

II. $(5\alpha + 60^\circ) + (-\alpha - 192^\circ) = 180^\circ + k_2 \cdot 360^\circ$: $4\alpha = 312^\circ + k_2 \cdot 360^\circ$, so $\alpha = 78^\circ + k_2 \cdot 90^\circ$.

In case I the solutions in the interval are $-162^\circ, -102^\circ, -42^\circ, 18^\circ, 78^\circ, 138^\circ$; in case II, $-102^\circ, -12^\circ, 78^\circ, 168^\circ$. So the solutions are

$$-162^\circ, -102^\circ, -42^\circ, -12^\circ, 18^\circ, 78^\circ, 138^\circ, 168^\circ$$

Note: In case II KoMaL writes $4\alpha = 132^\circ + k \cdot 360^\circ$ and lists $-147^\circ, -57^\circ, 33^\circ, 123^\circ$; these are not solutions (for example, $\sin 225^\circ \neq \sin(-225^\circ)$). The correct equation is $4\alpha - 132^\circ = 180^\circ + k \cdot 360^\circ$.

3. In a class of 32 students, the average height of the girls is 165 cm, and that of the boys is 172 cm.

a) Give the narrowest interval in which the average height of the class can fall.

b) During the year a 170 cm tall girl joined the class. The average height of the girls remained an integer. How many girls could there have been in the class originally?

(14 points)

Solution: a) The class has both boys and girls, since we can talk about their average heights. The average of the class is smallest if there is only one boy:

$$\frac{31 \cdot 165 + 172}{32} = 165.21875$$

and largest if there is only one girl:

$$\frac{165 + 31 \cdot 172}{32} = 171.78125$$

So the average height of the class falls in $[165.21875; 171.78125]$, and no narrower interval can be given.

b) Let the original number of girls be x . After the new girl arrives, the average height of the girls is

$$\frac{165x + 170}{x + 1} = \frac{165(x + 1) + 5}{x + 1} = 165 + \frac{5}{x + 1}$$

This is an integer, so $x + 1$ is a divisor of 5; since x is a positive integer, $x + 1 = 5$, $x = 4$. There were originally 4 girls in the class.

4. The edges of the rectangular base of a right prism are 3 and 5, and its space diagonal is $\frac{\sqrt{497}}{2}$.

a) What angle does the space diagonal make with the shorter base edge?

b) What is the surface area of the solid?

c) We want to design a square prism with the same height and the same volume as this prism. By what percent must the base edges be changed?

(14 points)

Solution: a) Let A be a vertex of the base, $AD = 3$ the shorter base edge, and G the opposite vertex of the prism. In the right triangle ADG (with the right angle at D),

$$\cos \alpha = \frac{AD}{AG} = \frac{3}{\frac{\sqrt{497}}{2}}, \quad \text{so} \quad \alpha \approx 74.4^\circ$$

b) Let the height of the prism be m . The space diagonal is $\sqrt{5^2 + 3^2 + m^2} = \frac{\sqrt{497}}{2}$, so $m = \frac{19}{2}$. The surface area is

$$A = 2(ab + am + bm) = 2\left(15 + \frac{95}{2} + \frac{57}{2}\right) = 182$$

c) Let the base edge of the square prism be x . From $3 \cdot 5 \cdot \frac{19}{2} = x^2 \cdot \frac{19}{2}$, $x = \sqrt{15}$. The edges must be multiplied by

$$p = \frac{\sqrt{15}}{5} \approx 0.775 \quad \text{and} \quad q = \frac{\sqrt{15}}{3} \approx 1.291$$

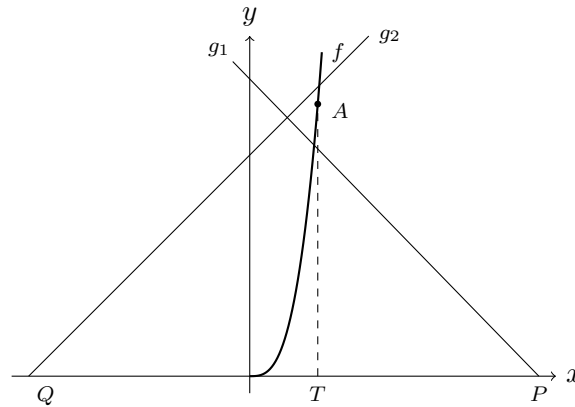
so the longer base edge must be decreased by about 22.5%, and the shorter one increased by about 29.1%.

Part II

5. The functions $f(x) = x^3$, defined on the nonnegative real numbers, and $g(x) = mx - 2m + 8$, defined on the real numbers, are given (m is a real parameter). The area of the region bounded by the graphs of the functions and the x -axis is 2010. Find the value of the parameter m .

(16 points)

Solution: $g(x) = m(x - 2) + 8$, so the point $A(2, 8)$ is on the graph of g for every m , and it is also on the graph of f . The line can be in two positions satisfying the conditions; let it meet the x -axis at $P(p, 0)$ ($p > 2$, the line g_1) or at $Q(-q, 0)$ ($q > 0$, the line g_2), and let $T(2, 0)$.



The area under the graph of f on $[0, 2]$ is $\int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 = 4$.

I. The region OAP : the right triangle ATP has area $\frac{(p-2) \cdot 8}{2} = 4(p-2)$, so $4(p-2) + 4 = 2010$, $p = 503.5$. From $0 = m(503.5 - 2) + 8$,

$$m_1 = -\frac{16}{1003} \approx -0.01596$$

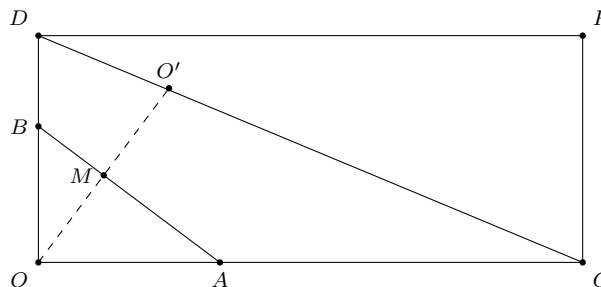
II. The region OAQ : the right triangle ATQ has area $\frac{(q+2) \cdot 8}{2} = 4(q+2)$, so $4(q+2) - 4 = 2010$, $q = 501.5$. From $0 = m(-501.5 - 2) + 8$,

$$m_2 = \frac{16}{1007} \approx 0.01589$$

6. From one corner of a rectangular sheet of paper of size 5 cm by 12 cm we measure 3 cm on the shorter side and 4 cm on the longer side. Along the line through the two points obtained in this way we would like to fold this corner of the sheet onto the diagonal opposite to the corner. Can this be done? Where does the corner go?

(16 points)

Solution: We place the rectangle in a coordinate system: its vertices are $O(0, 0)$, $C(12, 0)$, $P(12, 5)$, $D(0, 5)$. The points measured from the origin are $A(4, 0)$ and $B(0, 3)$. We fold the corner O along the line AB ; we need to find the reflection O' of O in AB .



The line AB is $3x + 4y = 12$, and the line through O perpendicular to it is $-4x + 3y = 0$. They meet at $M\left(\frac{36}{25}, \frac{48}{25}\right)$, and since M is the midpoint of OO' , $O'\left(\frac{72}{25}, \frac{96}{25}\right)$. The diagonal CD is $5x + 12y = 60$. Substituting the coordinates of O' :

$$5 \cdot \frac{72}{25} + 12 \cdot \frac{96}{25} = \frac{1512}{25} = 60.48 \neq 60$$

so O' is not on the diagonal CD . The point of the diagonal with first coordinate $\frac{72}{25}$ has second coordinate $\frac{95}{25}$, so O' is higher: after folding, the corner goes over to the other side of the diagonal. So it cannot be done.

7. The isosceles triangles $AB_1C_1, AB_2C_2, \dots, AB_nC_n, \dots$ (n a positive integer) are given. For every n , the base AB_n of the triangle lies on the positive half of the x -axis so that the vertex A is at the origin, and the base has length $2n$. We also know that the third vertex is on the graph of the function $f(x) = x^2 + 5$.

- Find the perimeter and the area of the triangle AB_1C_1 .
- Starting from which triangle is the perimeter greater than 190?
- Prove that the area of every triangle is divisible by six.

(16 points)

Solution: a) The vertices of the triangle are $A(0, 0)$, $B_1(2, 0)$, $C_1(1, 6)$. Since $AB_1 = 2$ and $AC_1 = B_1C_1 = \sqrt{37}$, the perimeter is $k_1 = 2 + 2\sqrt{37} \approx 14.17$. The altitude to AB_1 is 6, so the area is $t_1 = \frac{2 \cdot 6}{2} = 6$.

b) The vertices of the triangle AB_nC_n are $A(0, 0)$, $B_n(2n, 0)$, $C_n(n, n^2 + 5)$, so

$$AC_n = B_nC_n = \sqrt{n^2 + (n^2 + 5)^2} = \sqrt{n^4 + 11n^2 + 25}, \quad k_n = 2n + 2\sqrt{n^4 + 11n^2 + 25}$$

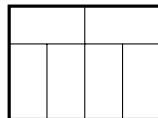
We look for the smallest positive n with $\sqrt{n^4 + 11n^2 + 25} > 95 - n$. The left side is increasing and the right side decreasing, so a few trials give the answer: for $n = 8$, $k_8 \approx 154.92 < 190$, and for $n = 9$, $k_9 \approx 190.94 > 190$. So starting from the ninth triangle, every perimeter is greater than 190.

c) The area of the triangle AB_nC_n is

$$t_n = \frac{2n(n^2 + 5)}{2} = n(n^2 + 5) = n^3 - n + 6n = (n - 1)n(n + 1) + 6n$$

The second term is divisible by 6. The first term is the product of three consecutive integers; one of them is divisible by 3, and at least one is even, so it is divisible by 6 too. So the area of every triangle is divisible by six.

8. The figure shows the shape of the windows of a school.



One class decided to put at most one glass sticker in the middle of each rectangular glass pane of the windows.

- In how many ways can the stickers be placed on a window if they plan four stickers on the six panes and the stickers are identical?
- In how many ways can the stickers be placed on a window if they plan three stickers on the six panes and the stickers are different?
- In how many ways can the stickers be placed if the panes with (identical) stickers are chosen symmetrically about the vertical axis of the window?
- All six parts of the window can be opened separately. They choose two at random (with equal probability) to open for ventilation. What is the probability that, for a window with three stickers, they open two undecorated parts?

(16 points)

Solution: a) Any four of the six panes can be chosen: $\binom{6}{4} = \binom{6}{2} = 15$ ways.

b) Any three of the six panes can be chosen, in $\binom{6}{3} = 20$ ways; since the stickers are different, they can be put on the chosen panes in 6 ways. So there are 120 ways.

c) The six panes are symmetric about a vertical axis, three on each side. If the decoration is symmetric, we decorate one half of the window in every possible way and the other half symmetrically. One half can have 1, 2, or 3 stickers: $3 + 3 + 1 = 7$ ways.

d) Two of the six parts can be chosen in 15 ways, all equally likely. Three parts have no sticker, and two of them can be chosen in 3 ways. So the probability is $\frac{3}{15} = \frac{1}{5}$.

9. The horizontal part of Main Square, in the shape of an equilateral triangle, is covered with decorative paving. In the middle of the triangle stands a tall flagpole. At each vertex of the triangle there is a person about 180 cm tall. Two of them start walking toward the flagpole. One stops when he sees the top of the pole at an angle of elevation of 72° . The other stops when he sees the top of the pole at 65° . Their companion who stayed in place sees the top of the flagpole at 50° . Then the area of the triangle determined by the three people is 23.74 m^2 .

a) How tall is the flagpole?

b) What is the area of the decorated part?

(16 points)

Solution: Let Z be the top of the flagpole and T_1 the point of the pole at eye level (1.8 m). Let A_1, B_1, C_1 be the eye-level points of the three people; T_1A_1, T_1B_1, T_1C_1 point toward the three vertices of the triangle, so they make angles of 120° with each other.

a) We know that $\angle ZA_1T_1 = 72^\circ, \angle ZB_1T_1 = 65^\circ, \angle ZC_1T_1 = 50^\circ$. Let $T_1Z = x$; then

$$T_1C_1 = \frac{x}{\tan 50^\circ}, \quad T_1B_1 = \frac{x}{\tan 65^\circ}, \quad T_1A_1 = \frac{x}{\tan 72^\circ}$$

The area of the triangle $A_1B_1C_1$ is the sum of the areas of the triangles $A_1T_1B_1, B_1T_1C_1, C_1T_1A_1$:

$$\frac{1}{2} \left(\frac{x^2}{\tan 72^\circ \tan 65^\circ} + \frac{x^2}{\tan 65^\circ \tan 50^\circ} + \frac{x^2}{\tan 50^\circ \tan 72^\circ} \right) \sin 120^\circ = 23.74$$

so $x \approx 8.2 \text{ m}$. Adding the eye height 1.8 m, the flagpole is about 10 m tall.

b) The person who stayed in place is at a vertex, so the distance of the vertices from the foot of the pole is $\frac{8.2}{\tan 50^\circ}$. Using the area formula with sine again,

$$t_{ABC} = 3 \cdot \frac{1}{2} \cdot \frac{8.2^2}{\tan^2 50^\circ} \cdot \sin 120^\circ \approx 61.5$$

The decorated part has an area of about 61.5 m^2 .