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translated from: KoMaL, March 2010, pages 142-143

Pdf file: KoMaL, March 2010, pages 142-143

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The arithmetic sequence (a_n) is given. Prove that the sequence defined by $b_n = a_{n+1}^2 - a_n^2$ is also an arithmetic sequence.

(11 points)

Solution: We show that the difference of any two neighboring terms of (b_n) is constant. Let the difference of (a_n) be d . Then

$$\begin{aligned} b_n &= (a_n + d)^2 - a_n^2 = 2a_nd + d^2 \\ b_{n+1} &= 2a_{n+1}d + d^2 = 2(a_n + d)d + d^2 = 2a_nd + 3d^2 \end{aligned}$$

So $b_{n+1} - b_n = 2d^2$, which is indeed constant.

2. Solve the following equation over the real numbers.

$$\frac{\sqrt{x+1} + 2}{\sqrt{x+1} - 1} = \frac{x+1}{x-2}$$

(12 points)

Solution: The domain of the equation is $x \geq -1$, but $x \neq 0$ and $x \neq 2$. Let $\sqrt{x+1} = a$; then the equation is

$$\frac{a+2}{a-1} = \frac{a^2}{a^2-3}$$

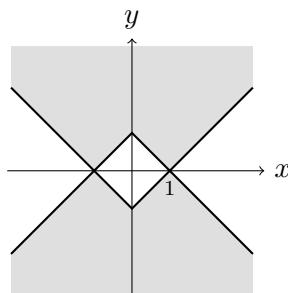
which gives $a^2 - a - 2 = 0$, so $a_1 = -1$ and $a_2 = 2$. $\sqrt{x+1} = -1$ is impossible, and $\sqrt{x+1} = 2$ gives the only solution, $x = 3$.

3. Which are the points $P(x, y)$ for which $||x| - 1| - |y| \leq 0$?

(14 points)

Solution: Rearranging the inequality: $||x| - 1| \leq |y|$.

If $y \geq 0$, then $||x| - 1| \leq y$: the points on and above the graph of $y = ||x| - 1|$. If $y < 0$, then $||x| - 1| \leq -y$, that is, $-||x| - 1| \geq y$: the points on and below the graph of $y = -||x| - 1|$. The union of the two cases gives the solution: the shaded region (with its boundary) in the figure.



4. An urn contains 7 red and 9 blue balls. We draw five balls one after another, putting the drawn ball back after each draw. What is the probability that there are more red balls than blue ones among the drawn balls?

(14 points)

Solution: The number of all cases is 16^5 . Three kinds of cases are favorable:

I. 3 red and 2 blue balls: $\frac{5!}{3! \cdot 2!} \cdot 7^3 \cdot 9^2$ cases.

II. 4 red and 1 blue ball: $\frac{5!}{4! \cdot 1!} \cdot 7^4 \cdot 9$ cases.

III. 5 red balls: 7^5 cases.

So the probability is

$$\frac{\frac{5!}{3! \cdot 2!} \cdot 7^3 \cdot 9^2 + \frac{5!}{4! \cdot 1!} \cdot 7^4 \cdot 9 + 7^5}{16^5} \approx 0.3840$$

Part II

5. A telecommunications company has 13 000 subscribers with a monthly fee of 6500 Ft. Market research shows that if the monthly fee were decreased by 100 Ft, the company would get 250 new subscribers, and this would be true for every further decrease by 100 Ft. (The monthly fee at this company is always divisible by 100.) With what monthly fee would the company have the largest revenue, according to the market research?

(16 points)

Solution: The revenue of the company is $f(x) = (6500 - 100x)(13\,000 + 250x)$, where $0 \leq x < 65$ and $x \in \mathbb{Z}$. This can be written as

$$f(x) = -25\,000(x - 65)(x + 52)$$

The quadratic function has its maximum at $x = 6.5$. Since x can only be an integer, and the graph is symmetric about the axis $x = 6.5$, the function has equal values at $x_1 = 6$ and $x_2 = 7$. So the most favorable monthly fee for the company would be 5900 Ft or 5800 Ft (the revenue is 85 550 000 Ft in both cases).

6. Let A_1, B_1, C_1 be arbitrary points on the sides BC, CA, AB of the triangle ABC . Let $l_a = AA_1, l_b = BB_1, l_c = CC_1$. Prove that

$$\frac{1}{2} < \frac{l_a + l_b + l_c}{a + b + c} < \frac{3}{2}$$

(16 points)

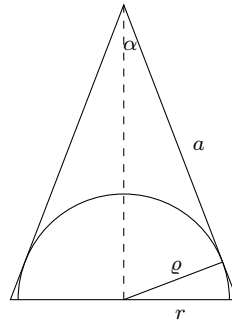
Solution: We use the triangle inequality: $l_a > b - A_1C$ and $l_a > c - A_1B$. Adding them, $2l_a > b + c - a$. In the same way, $2l_b > a + c - b$ and $2l_c > a + b - c$. Adding the three inequalities and dividing by 2: $l_a + l_b + l_c > \frac{1}{2}(a + b + c)$, which is one of the inequalities to prove.

Using the triangle inequality again: $l_a < c + A_1B$ and $l_a < b + A_1C$. Adding them, $2l_a < a + b + c$. In the same way, $2l_b < a + b + c$ and $2l_c < a + b + c$. Adding the three inequalities and dividing by 2: $l_a + l_b + l_c < \frac{3}{2}(a + b + c)$, which is the other inequality.

7. We inscribe a hemisphere in a right circular cone so that its circular face lies in the plane of the base circle of the cone, and its spherical surface touches the lateral surface of the cone. The surface area of the cone is to the area of the curved surface of the hemisphere as 18 : 5. What is the vertex angle of the cone?

(16 points)

Solution: Consider the cross section through the axis of the cone. Let r be the radius of the base, a the slant height, ϱ the radius of the hemisphere, and α half the vertex angle. Then $a = \frac{r}{\sin \alpha}$ and $\varrho = r \cos \alpha$, where $0 < \alpha < 90^\circ$.



Let A be the surface area of the cone and B the area of the curved surface of the hemisphere:

$$\frac{A}{B} = \frac{r \left(\frac{r}{\sin \alpha} + r \right) \pi}{2r^2 \pi \cos^2 \alpha} = \frac{18}{5}, \quad \text{that is,} \quad \frac{1 + \sin \alpha}{2 \sin \alpha \cos^2 \alpha} = \frac{18}{5}$$

Using $\cos^2 \alpha = 1 - \sin^2 \alpha$ and $\sin \alpha + 1 \neq 0$:

$$\frac{1}{2 \sin \alpha (1 - \sin \alpha)} = \frac{18}{5}, \quad 36 \sin^2 \alpha - 36 \sin \alpha + 5 = 0$$

so $\sin \alpha = \frac{5}{6}$ or $\sin \alpha = \frac{1}{6}$. The vertex angle of the cone is $2\alpha \approx 112.89^\circ$ or $2\alpha \approx 19.19^\circ$.

8. Bálint and Jonatán play the following game. They roll two dice; if the product or the sum of the numbers rolled is divisible by three, Bálint wins, otherwise Jonatán wins the game. Who has the greater chance of winning?

(16 points)

Solution: Let A be the event that the product of the two numbers is divisible by three, and B the event that their sum is divisible by three. We know that $P(A + B) = P(A) + P(B) - P(AB)$.

The sum is divisible by 3 in 12 of the 36 cases, so $P(B) = \frac{12}{36} = \frac{1}{3}$. The product is divisible by 3 if at least one number is 3 or 6, in $36 - 16 = 20$ cases, so $P(A) = \frac{20}{36} = \frac{5}{9}$. Both happen only if both numbers are divisible by 3, in 4 cases, so $P(AB) = \frac{4}{36} = \frac{1}{9}$.

Bálint's chance of winning is $P(A + B) = \frac{3+5-1}{9} = \frac{7}{9}$, and Jonatán's is $1 - \frac{7}{9} = \frac{2}{9}$. So Bálint has the greater chance of winning.

9. For which real numbers p does the following inequality hold for every real number x ?

$$\frac{2x^2 + 2x + 3}{x^2 + x + 1} \leq p$$

(16 points)

Solution: Since $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$, the denominator is positive for every real x . Multiplying by it:

$$0 \leq (p - 2)x^2 + (p - 2)x + p - 3$$

If $p = 2$, the inequality is $0 \leq -1$, which holds for no real x . Otherwise $p > 2$ is needed (a positive leading coefficient), and the discriminant must not be positive:

$$(p - 2)^2 - 4(p - 2)(p - 3) \leq 0, \quad \text{that is,} \quad (p - 2)(-3p + 10) \leq 0$$

Since $p > 2$, this holds if and only if $p \geq \frac{10}{3}$. (Indeed, the largest value of the left side of the original inequality is $\frac{10}{3}$, at $x = -\frac{1}{2}$.)