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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. At the carnival raffle of an elementary school, a total of 210 raffle tickets were sold, and Bori bought three of them. At the drawing, 20 winning tickets are drawn, and the owner of the last one drawn wins the grand prize. Find the probability that
 - a) Bori wins with at least one of her tickets;
 - b) Bori wins with all three of her tickets;
 - c) Bori wins the grand prize.

(12 points)

2. Let P be a point in the coordinate plane such that the tangent segments drawn from P to the circles $x^2 + y^2 = 6y - 6$ and $x^2 + y^2 = 2x$ (from P to the points of tangency) are all equally long. Prove that the set of points P with this property is a line, and find the equation of this line.

(13 points)

3. Find the values of the parameters p and q of the function $f(x) = 3x^2 + px + q$, if

$$\int_1^3 f(x) dx = 13 \quad \text{and} \quad \int_2^4 f(x) dx = 35$$

(12 points)

4. a) Find the largest subset of the real numbers on which the function

$$x \mapsto \frac{\sin 2x}{\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right)}$$

is defined, and find the range of the function on this domain. Graph the function on the interval $[-2\pi, 2\pi]$.

- b) Solve the equation

$$\frac{\sin 2x}{\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right)} = 2 \cos 2x$$

over the real numbers.

(14 points)

Part II

5. The sides of an equilateral triangle with side length 1 unit are divided in the ratio $m : n$ (all in the same direction around the triangle), and connecting the division points we get a new triangle. Imagine repeating this process infinitely many times with each triangle obtained. The sum of the areas of the triangles obtained is $\frac{19}{30}$ times the area of the original triangle. Find the ratio $m : n$.

(16 points)

6. a) Find the value of x so that $16 \cdot 25^x$, $133 \cdot 10^x$, and $250 \cdot 4^x$ (in this order) are consecutive terms of an arithmetic sequence.
- b) The first term of an arithmetic sequence is 16, and its common difference is 117. The sum of the first n terms of the sequence is 371 000. Find the value of n .

(16 points)

7. All edges of a regular pentagonal pyramid are equal.

- a) What is the angle between two adjacent lateral faces?
- b) What is the angle between a lateral edge and the base?
- c) A lead paperweight has exactly the shape of such a regular pentagonal pyramid, with all edges 1.5 cm long. How many paperweights can be cast from 1.5 kg of lead, if experience shows that we must count on a 9% loss in the process? (The density of lead is 11.3 g/cm^3 .)

(16 points)

8. The four resort towns of the largest lake of Bergengocia are to be connected by boat routes. Each route connects two towns directly, without stopping at a third town (the boats run in both directions). Any number of routes from 0 to 6 may operate in the network.

- a) How many different route networks can be formed? (Two networks are different if there is at least one route that operates in one network but not in the other.)
- b) What fraction of all possible route networks are connected? (A network is connected if one can get from any town to any other town, changing boats if necessary.)
- c) To save money, they finally decide to operate the fewest possible routes while the network is still connected. How many different route networks can be formed this way?

(16 points)

9. A 180 cm high fence separates Mr. Murmur's lot from his neighbor's lot. At what angle should he lean his 390 cm tall folding screen against the fence, if he wants it to cast the largest possible shadow onto the neighbor's garden, assuming that the Sun is shining exactly vertically? How many meters does this shadow reach into the neighbor's lot?

(16 points)