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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. At the carnival raffle of an elementary school, a total of 210 raffle tickets were sold, and Bori bought three of them. At the drawing, 20 winning tickets are drawn, and the owner of the last one drawn wins the grand prize. Find the probability that
 - a) Bori wins with at least one of her tickets;
 - b) Bori wins with all three of her tickets;
 - c) Bori wins the grand prize.

(12 points)

Solution: a) We compute the probability of the complementary event and subtract it from 1. If Bori did not win, then all 20 winning tickets were drawn from the 207 tickets she did not buy. So

$$P(\text{Bori wins}) = 1 - \frac{\binom{207}{20}}{\binom{210}{20}} = 1 - \frac{207 \cdot 206 \cdot \dots \cdot 188}{210 \cdot 209 \cdot \dots \cdot 191} = 1 - \frac{190 \cdot 189 \cdot 188}{210 \cdot 209 \cdot 208} \approx 0.2605$$

- b) If Bori won with all three of her tickets, then the other 17 winning tickets were drawn from the other 207 tickets:

$$P(\text{Bori wins with all three}) = \frac{\binom{3}{3} \binom{207}{17}}{\binom{210}{20}} = \frac{20 \cdot 19 \cdot 18}{210 \cdot 209 \cdot 208} \approx 0.00075$$

- c) Each of the 210 tickets sold is equally likely to win the grand prize, and three of them are Bori's, so

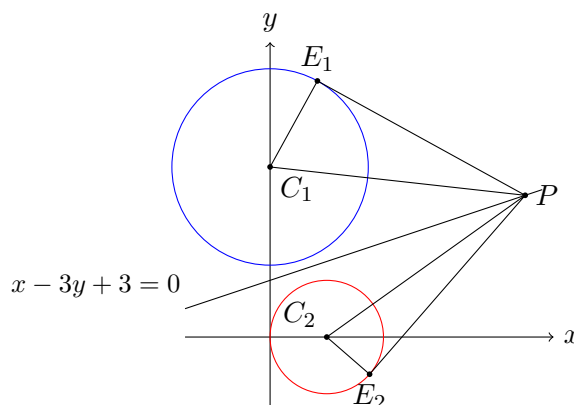
$$P(\text{Bori wins the grand prize}) = \frac{3}{210} = \frac{1}{70} \approx 0.014$$

2. Let P be a point in the coordinate plane such that the tangent segments drawn from P to the circles $x^2 + y^2 = 6y - 6$ and $x^2 + y^2 = 2x$ (from P to the points of tangency) are all equally long. Prove that the set of points P with this property is a line, and find the equation of this line.

(13 points)

Solution: Rewrite the equations of the circles: $x^2 + (y - 3)^2 = 3$, so the center of the first circle is $C_1(0, 3)$ and its radius is $r_1 = \sqrt{3}$; $(x - 1)^2 + y^2 = 1$, so the center of the second circle is $C_2(1, 0)$ and its radius is $r_2 = 1$.

If we draw a tangent line from an outside point P to a circle, then the center of the circle, the point of tangency, and P form a right triangle (with the right angle at the point of tangency). Let E_1 and E_2 be points of tangency on the two circles. Then $PE_1 = \sqrt{PC_1^2 - r_1^2}$ and $PE_2 = \sqrt{PC_2^2 - r_2^2}$, so $PE_1 = PE_2$ means $PC_1^2 - r_1^2 = PC_2^2 - r_2^2$.



The distances of the point $P(x, y)$ from the centers are $PC_1 = \sqrt{x^2 + (y - 3)^2}$ and $PC_2 = \sqrt{(x - 1)^2 + y^2}$. Substituting these:

$$x^2 + (y - 3)^2 - 3 = (x - 1)^2 + y^2 - 1$$

After rearranging, the quadratic terms cancel, and we get $x - 3y + 3 = 0$. So the points with the property lie on this line. Conversely, every point of this line has the property, since the steps above can be reversed. For this we only need to check that $PC_i^2 - r_i^2 \geq 0$ for every point P of the line; indeed, with $x = 3y - 3$,

$$PC_1^2 - r_1^2 = (3y - 3)^2 + (y - 3)^2 - 3 = 10y^2 - 24y + 15$$

which is always positive, since its discriminant is negative.

Remark: The line obtained is the radical axis of the two circles; its equation is the difference of the equations of the two circles.

3. Find the values of the parameters p and q of the function $f(x) = 3x^2 + px + q$, if

$$\int_1^3 f(x) dx = 13 \quad \text{and} \quad \int_2^4 f(x) dx = 35$$

(12 points)

Solution: Each integral gives an equation for the two unknown parameters.

$$\int_1^3 (3x^2 + px + q) dx = \left[x^3 + \frac{px^2}{2} + qx \right]_1^3 = (27 + 4.5p + 3q) - (1 + 0.5p + q) = 26 + 4p + 2q = 13$$

$$\int_2^4 (3x^2 + px + q) dx = \left[x^3 + \frac{px^2}{2} + qx \right]_2^4 = (64 + 8p + 4q) - (8 + 2p + 2q) = 56 + 6p + 2q = 35$$

So $4p + 2q = -13$ and $6p + 2q = -21$. The solution of this system is $p = -4$, $q = 1.5$, so $f(x) = 3x^2 - 4x + 1.5$.

4. a) Find the largest subset of the real numbers on which the function

$$x \mapsto \frac{\sin 2x}{\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right)}$$

is defined, and find the range of the function on this domain. Graph the function on the interval $[-2\pi, 2\pi]$.

- b) Solve the equation

$$\frac{\sin 2x}{\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right)} = 2 \cos 2x$$

over the real numbers.

(14 points)

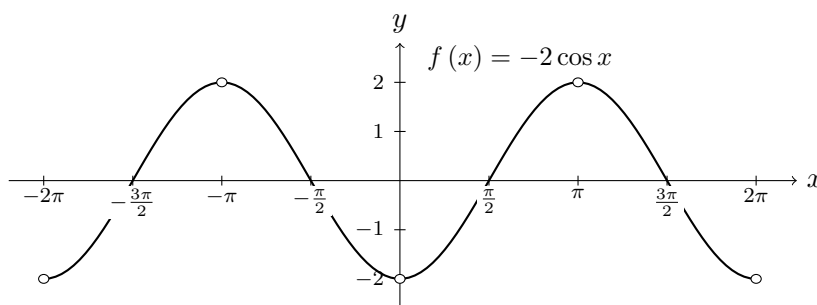
Solution: a) We transform the denominator:

$$\cos\left(x + \frac{\pi}{6}\right) - \cos\left(x - \frac{\pi}{6}\right) = \left(\cos x \cos \frac{\pi}{6} - \sin x \sin \frac{\pi}{6}\right) - \left(\cos x \cos \frac{\pi}{6} + \sin x \sin \frac{\pi}{6}\right) = -2 \sin x \cdot \frac{1}{2} = -\sin x$$

So the function is

$$x \mapsto \frac{2 \sin x \cos x}{-\sin x} = -2 \cos x$$

where $\sin x \neq 0$. So the domain of the function is $\mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$. Since $\cos x = \pm 1$ exactly at the excluded points, the range of the function is the open interval $(-2, 2)$. The figure shows the graph of the function on the given interval.



- b) Using the previous result and well-known trigonometric identities, the equation can be transformed as follows (where $\sin x \neq 0$):

$$-\cos x = \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1, \quad \text{so} \quad 2 \cos^2 x + \cos x - 1 = 0$$

This equation is quadratic in $\cos x$; its solutions are $\cos x = \frac{1}{2}$ and $\cos x = -1$. In the first case $x = \pm \frac{\pi}{3} + 2k\pi$ ($k \in \mathbb{Z}$); in the second case $\sin x = 0$ would hold, which contradicts the condition, so we get no solution from there. Our transformations were equivalent, so the solutions are

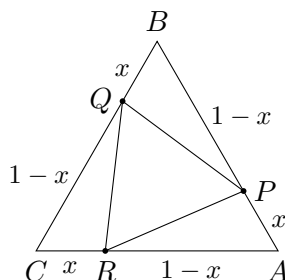
$$x = \pm \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

Part II

5. The sides of an equilateral triangle with side length 1 unit are divided in the ratio $m : n$ (all in the same direction around the triangle), and connecting the division points we get a new triangle. Imagine repeating this process infinitely many times with each triangle obtained. The sum of the areas of the triangles obtained is $\frac{19}{30}$ times the area of the original triangle. Find the ratio $m : n$.

(16 points)

Solution: On the sides AB , BC , and CA of the equilateral triangle ABC with side 1, let P , Q , and R be the points with $AP = BQ = CR = x$ and $PB = QC = RA = 1 - x$. So $m : n = x : (1 - x)$.



The triangle PQR is also equilateral, since the triangles APR , BQP , and CRQ are congruent (two sides and the angle between them are equal). So every triangle obtained in the process is equilateral, and by similarity (each time the sides are divided in the ratio $m : n$), the sides of the triangles form a geometric sequence with ratio $q = \frac{PQ}{AB}$. The ratio of the areas of similar figures is the square of the ratio of similarity, so the areas form a geometric sequence with ratio $q^2 < 1$.

Let a_1 be the area of the original triangle. The sum of the areas of the new triangles is

$$\frac{a_1 q^2}{1 - q^2} = \frac{19}{30} a_1, \quad \text{so} \quad q^2 = \frac{19}{49} \quad \text{and} \quad q = \frac{\sqrt{19}}{7}$$

($q < 0$ makes no sense here). So $PR = \frac{\sqrt{19}}{7}$. By the law of cosines in the triangle APR ,

$$\left(\frac{\sqrt{19}}{7}\right)^2 = x^2 + (1 - x)^2 - 2x(1 - x)\cos 60^\circ$$

Expanding, multiplying by 49 and rearranging: $49x^2 - 49x + 10 = 0$. The roots of the equation are $x = \frac{2}{7}$ and $x = \frac{5}{7}$. So

$$m : n = x : (1 - x) = 2 : 5 \quad (\text{or } 5 : 2)$$

6. a) Find the value of x so that $16 \cdot 25^x$, $133 \cdot 10^x$, and $250 \cdot 4^x$ (in this order) are consecutive terms of an arithmetic sequence.
- b) The first term of an arithmetic sequence is 16, and its common difference is 117. The sum of the first n terms of the sequence is 371 000. Find the value of n .

(16 points)

Solution: a) By the conditions, $133 \cdot 10^x - 16 \cdot 25^x = 250 \cdot 4^x - 133 \cdot 10^x$, that is, $16 \cdot 25^x - 266 \cdot 10^x + 250 \cdot 4^x = 0$. Divide the equation by 4^x , which is positive:

$$16 \cdot \frac{25^x}{4^x} - 266 \cdot \frac{10^x}{4^x} + 250 = 16 \left(\frac{5}{2}\right)^{2x} - 266 \left(\frac{5}{2}\right)^x + 250 = 0$$

This equation is quadratic in $\left(\frac{5}{2}\right)^x$; its solutions are $\frac{125}{8}$ and 1. In the first case $x = 3$, and in the second case $x = 0$. In both cases we indeed get an arithmetic sequence: 250 000, 133 000, 16 000, and 16, 133, 250.

b) By the formula for the sum of an arithmetic sequence,

$$S_n = \frac{[2a_1 + (n-1)d]n}{2} = \frac{[32 + 117(n-1)]n}{2} = 371\,000$$

Multiplying by 2 and rearranging, $117n^2 - 85n - 742\,000 = 0$. Its positive solution is $n = 80$ (the other root is $-\frac{9275}{117} \approx -79.27$). So the sum of the first 80 terms of the sequence is 371 000.

7. All edges of a regular pentagonal pyramid are equal.

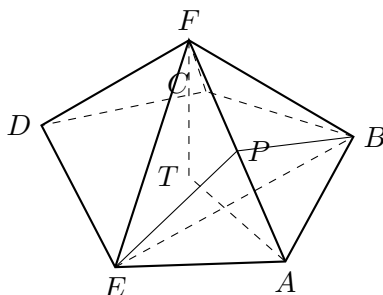
a) What is the angle between two adjacent lateral faces?

b) What is the angle between a lateral edge and the base?

c) A lead paperweight has exactly the shape of such a regular pentagonal pyramid, with all edges 1.5 cm long. How many paperweights can be cast from 1.5 kg of lead, if experience shows that we must count on a 9% loss in the process? (The density of lead is 11.3 g/cm^3 .)

(16 points)

Solution: a) Let the base of the pyramid be the regular pentagon $ABCDE$, let its apex be F , and let its edge length be a . The angle of two lateral faces is the angle of the perpendiculars to their common edge drawn in the planes of the two faces. The perpendiculars to the edge AF at its midpoint P in the faces AFE and ABF are the altitudes of these equilateral triangles, so $PE = PB = \frac{\sqrt{3}}{2}a$.



The angle EAB is an interior angle of the regular pentagon, so it is 108° . By the law of cosines in the triangle EAB ,

$$BE^2 = a^2 + a^2 - 2a^2 \cos 108^\circ \approx 2.618a^2, \quad \text{so} \quad BE \approx 1.618a$$

(It is known that the diagonal and the side of a regular pentagon are in the golden ratio, so $BE = \frac{\sqrt{5}+1}{2}a$ exactly.) By the law of cosines in the triangle EBP ,

$$\cos \angle BPE = \frac{2\left(\frac{\sqrt{3}}{2}a\right)^2 - \left(\frac{\sqrt{5}+1}{2}a\right)^2}{2\left(\frac{\sqrt{3}}{2}a\right)^2} = -\frac{\sqrt{5}}{3} \approx -0.7454$$

so $\angle BPE \approx 138.19^\circ$. The angle between two adjacent lateral faces is about 138.19° .

b) Let T be the orthogonal projection of the apex onto the base; the angle in question is the angle TAF . The length AT can be computed from the isosceles triangle EAT , whose base angles are 54° :

$$\frac{a/2}{AT} = \cos 54^\circ, \quad \text{so} \quad AT = \frac{a}{2 \cos 54^\circ} \approx 0.8507a, \quad \text{and} \quad \cos \angle TAF = \frac{AT}{AF} \approx 0.8507$$

so $\angle TAF \approx 31.72^\circ$. The angle between a lateral edge and the base is about 31.72° .

c) The base of the pyramid is a regular pentagon, which the segments from its center to its vertices divide into five congruent isosceles triangles with legs $AT \approx 0.8507a$ and angle 72° between them. The height of the pyramid, by the Pythagorean theorem in the triangle AFT , is

$$m = TF = \sqrt{AF^2 - AT^2} \approx \sqrt{a^2 - (0.8507a)^2} \approx 0.5257a$$

If $a = 1.5$ cm, then the volume of the pyramid is

$$V = \frac{1}{3} \cdot 5 \cdot \frac{AT^2 \sin 72^\circ}{2} \cdot m \approx \frac{5 \sin 72^\circ \cdot 0.8507^2 \cdot 0.5257 \cdot a^3}{6} \approx 1.018 \text{ cm}^3$$

The volume of 1.5 kg = 1500 g of lead is $\frac{1500}{11.3} \approx 132.74 \text{ cm}^3$. With a 9% loss, the usable volume is $132.74 \cdot 0.91 \approx 120.80 \text{ cm}^3$. The number of paperweights that can be made from this much lead is at most $\frac{120.80}{1.018} \approx 118.7$, so 118 paperweights can be made from 1.5 kg of lead.

8. The four resort towns of the largest lake of Bergengocia are to be connected by boat routes. Each route connects two towns directly, without stopping at a third town (the boats run in both directions). Any number of routes from 0 to 6 may operate in the network.
- How many different route networks can be formed? (Two networks are different if there is at least one route that operates in one network but not in the other.)
 - What fraction of all possible route networks are connected? (A network is connected if one can get from any town to any other town, changing boats if necessary.)
 - To save money, they finally decide to operate the fewest possible routes while the network is still connected. How many different route networks can be formed this way?

(16 points)

Solution: a) Between the four towns $\binom{4}{2} = 6$ different routes can operate. Independently of each other, each of them either belongs to the network or not, so $2^6 = 64$ different route networks can be formed.

Remark: We get the same value by adding up the possible networks by the number of routes: $\binom{6}{0} + \binom{6}{1} + \cdots + \binom{6}{6} = 64$.

b) Consider the network as a graph, where the towns are the vertices and the routes are the edges. We are looking for connected simple graphs with 4 vertices.

Every graph with 4 vertices and at least 4 edges is connected; their number is

$$\binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 22$$

and the graphs with at most 2 edges are certainly not connected. There are $\binom{6}{3} = 20$ graphs with 3 edges; of these, the ones that are not connected are the triangles connecting 3 towns. There are 4 of these, so there are 16 connected graphs with 3 edges.

These are of two types. In the first type all 3 edges start from the same vertex; there are 4 of these, depending on which vertex the 3 edges start from. The second type is a path through all four vertices: the first vertex can be chosen in 4 ways, the second in 3, the third in 2, but we get the same graph if we list it from the other end. So there are $\frac{4 \cdot 3 \cdot 2}{2} = 12$ of these, and indeed $4 + 12 = 16$.

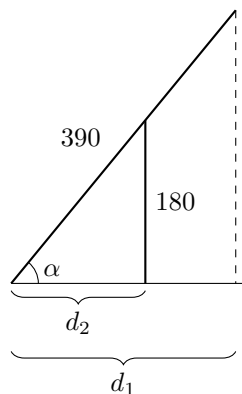
So the number of connected route networks is $22 + 16 = 38$, and $\frac{38}{64} = \frac{19}{32}$ of the networks are connected.

c) A connected network needs at least 3 routes, so the networks with exactly 3 routes are the ones we are looking for. By the above, their number is $4 + 12 = 16$.

9. A 180 cm high fence separates Mr. Murmur's lot from his neighbor's lot. At what angle should he lean his 390 cm tall folding screen against the fence, if he wants it to cast the largest possible shadow onto the neighbor's garden, assuming that the Sun is shining exactly vertically? How many meters does this shadow reach into the neighbor's lot?

(16 points)

Solution: Let α be the angle between the screen and the horizontal plane of the garden. Let d_1 be the whole shadow cast by the screen; the part d_2 of it falls into Mr. Murmur's garden, where $\cos \alpha = \frac{d_1}{390}$ and $\cot \alpha = \frac{d_2}{180}$.



So the shadow falling into the neighbor's garden is given by the function

$$f(\alpha) = d_1 - d_2 = 390 \cos \alpha - 180 \cot \alpha \quad (0^\circ < \alpha < 90^\circ)$$

It can have an extreme value where its derivative is zero:

$$f'(\alpha) = -390 \sin \alpha + \frac{180}{\sin^2 \alpha} = 0, \quad \text{so} \quad \sin \alpha = \sqrt[3]{\frac{180}{390}} \approx 0.7728$$

and $\alpha \approx 50.6^\circ$. The second derivative of f is

$$f''(\alpha) = -390 \cos \alpha - \frac{360 \cos \alpha}{\sin^3 \alpha}$$

and since $f''(50.6^\circ) < 0$, this is indeed a maximum. Then $f(50.6^\circ) \approx 99.7$ cm.

So Mr. Murmur should lean his screen against the fence at an angle of about 50.6° with the ground (about 39.4° with the fence), and then the shadow reaches about 1 meter into the neighbor's lot.