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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

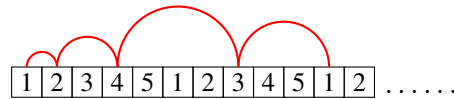
1. The figure shows a track consisting of 2011 squares. The numbers 1 to 5 were written into the squares, one after the other, repeatedly to the end of the track.

1 2 3 4 5 1 2 3 4 5 1 2

A piece stands on the first square. Each time, we move the piece forward as many squares as the number on the square where the piece currently stands. Find the sum of the numbers on the squares where the piece never lands while it goes along the track.

(10 points)

Solution: The figure shows the first 4 moves of the piece. With the fourth move the piece lands on a square with 1 again, so from then on the moves repeat periodically.



So one period consists of exactly 10 squares, and every fourth move starts a new period. Since $2011 = 10 \cdot 201 + 1$, there are 201 periods on the track, and with its last move the piece arrives at the last square of the track, which contains a 1.

In each period the piece does not land on the numbers 3, 5, 1, 2, 4, 5. So the sum of the numbers where the piece never lands is

$$201 \cdot (3 + 5 + 1 + 2 + 4 + 5) = 201 \cdot 20 = 4020$$

2. A large company has 1320 employees, and 60% of them eat in the company cafeteria. The ratio of the number of cafeteria users to the number of employees is $\frac{6}{5}$ times the ratio of the men using the cafeteria to the men employed by the company. The ratio of the men using the cafeteria to the men employed by the company and the ratio of the women using the cafeteria to the women employed by the company are in the ratio 2 : 3. How many men and how many women work at this company?

(13 points)

Solution: Let f and n be the numbers of men and women employed by the company, and let f_m and n_m be the numbers of men and women using the cafeteria. The number of cafeteria users is $1320 \cdot 0.6 = 792$, so $f + n = 1320$ and $f_m + n_m = 792$. Furthermore,

$$\frac{792}{1320} = \frac{6}{5} \cdot \frac{f_m}{f}, \quad \text{so} \quad f = 2f_m, \quad \text{and} \quad \frac{f_m}{f} : \frac{n_m}{n} = 2 : 3, \quad \text{so} \quad \frac{1}{2} = \frac{2}{3} \cdot \frac{n_m}{n}, \quad \text{and} \quad n = \frac{4}{3}n_m$$

So $1320 = f + n = 2f_m + \frac{4}{3}n_m = 2(f_m + n_m) - \frac{2}{3}n_m = 1584 - \frac{2}{3}n_m$, so $n_m = 396$ and $f_m = 792 - 396 = 396$. Then $f = 2f_m = 792$ and $n = \frac{4}{3}n_m = 528$.

So the company employs 792 men and 528 women.

3. 13 large boxes are stored in a warehouse. Some of these boxes each contain 13 medium-sized boxes. Some of the medium-sized boxes each contain 13 smaller boxes. The number of empty boxes is 205. How many boxes are there in the warehouse?

(14 points)

Solution: Let k of the 13 large boxes contain 13 medium-sized boxes each; then $13 - k$ large boxes are empty. Let n of the $13k$ medium-sized boxes contain 13 small boxes each. Then $13k - n$ medium-sized boxes are empty, and all $13n$ small boxes are empty too. So the number of empty boxes is

$$(13 - k) + (13k - n) + 13n = 205, \quad \text{so} \quad 12k + 12n = 192, \quad \text{and} \quad k + n = 16$$

The total number of boxes is $13 + 13k + 13n = 13(k + n + 1) = 13 \cdot 17 = 221$.

4. We place red and black balls into four boxes as shown in the table.

box	1	2	3	4
red	5	10	7	14
black	5	5	14	14

We draw one ball at random from each box. Prove that the probability that all four balls drawn are red is the same as the probability that all four balls are black.

(14 points)

Solution: The probabilities of drawing a red ball from the four boxes are $\frac{5}{10} = \frac{1}{2}$, $\frac{10}{15} = \frac{2}{3}$, $\frac{7}{21} = \frac{1}{3}$, and $\frac{14}{28} = \frac{1}{2}$. So the probability that all four balls are red is

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}$$

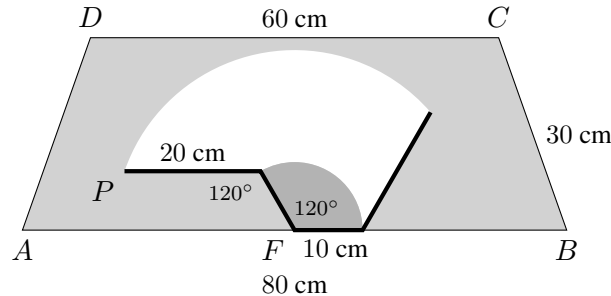
The probabilities of drawing a black ball from the boxes are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{2}$, so the probability that all four balls are black is

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{18}$$

The two probabilities are indeed equal.

Part II

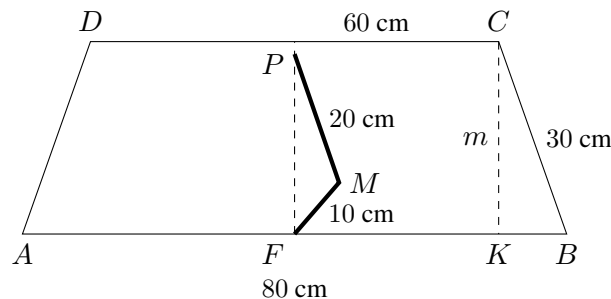
5. The figure shows the rear window of a car and its wiper. The wiper is an arm made of a 10 cm and a 20 cm piece joined at an angle of 120° ; the 10 cm piece is the crank, and the 20 cm piece is the rubber blade. The crank can turn through an angle of 120° about the midpoint F of the longer base of the rear window, which is a symmetric trapezoid. (Consider the rear window a plane figure.)



- a) How far is the endpoint P of the blade from the side CD of the window when it is closest to it?
 b) What percent of the rear window does the blade wipe?

(16 points)

Solution: a) Let M be the joint of the crank and the blade. The endpoint P of the blade is closest to the side CD when the line FP is perpendicular to the bases. (This happens during the motion: FP turns from 160.9° to 40.9° , measured from FB .) The distance in question is the difference of the height m of the trapezoid and the length of FP .



If K is the foot of the perpendicular from C to AB , then $BK = \frac{80-60}{2} = 10$ cm, so in the right triangle BCK , $m = \sqrt{30^2 - 10^2} = \sqrt{800} \approx 28.28$ cm. By the law of cosines in the triangle FMP ,

$$FP^2 = FM^2 + MP^2 - 2 \cdot FM \cdot MP \cos 120^\circ = 100 + 400 + 200 = 700, \quad FP = \sqrt{700} \approx 26.46 \text{ cm}$$

So the smallest distance of the endpoint of the blade from the side CD is $\sqrt{800} - \sqrt{700} \approx 1.83$ cm.

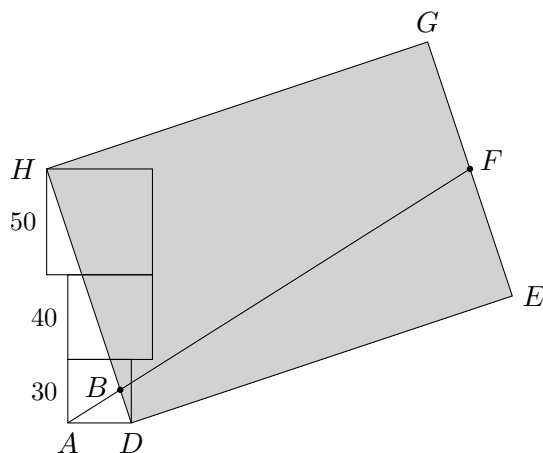
Note: KoMaL subtracts the rounded values and gets 1.82 cm.

b) The area of the trapezoid is $\frac{(80+60) \cdot \sqrt{800}}{2} \approx 1979.9 \text{ cm}^2$. In the first figure, M^* and Q are the final positions of M and P . The distances of the points of the blade from F increase from $FM = 10$ cm to $FP = 10\sqrt{7}$ cm, since the angle at M is obtuse. The wiped region is the sector PFQ of radius FP , plus the triangle FM^*Q , minus the triangle PFM and the sector $MFMM^*$ of radius 10 cm. The triangles FM^*Q and PFM are congruent, so the wiped area is

$$T = \frac{120^\circ}{360^\circ} \cdot (10\sqrt{7})^2 \pi - \frac{120^\circ}{360^\circ} \cdot 10^2 \pi = \frac{700\pi}{3} - \frac{100\pi}{3} = 200\pi \approx 628.3 \text{ cm}^2$$

So the blade wipes about $\frac{628.3}{1979.9} \cdot 100 \approx 31.7\%$ of the rear window.

6. The figure shows the floor plan of a block of buildings to be built and its underground garage. The floor plan of the block is three squares placed next to each other, with sides 50 m, 40 m, and 30 m. The underground garage is a rectangle $DEGH$ whose longer side DE is one and a half times its shorter side.



- a) What is the area of the underground garage?
- b) The car entrance of the garage will be at the midpoint F of the side EG . At the corner A of the block there will be a stairway down to the basement, from which a door at the point B leads into the garage. (A , B , and F are on a line, as in the figure.) How long is the segment BF ?

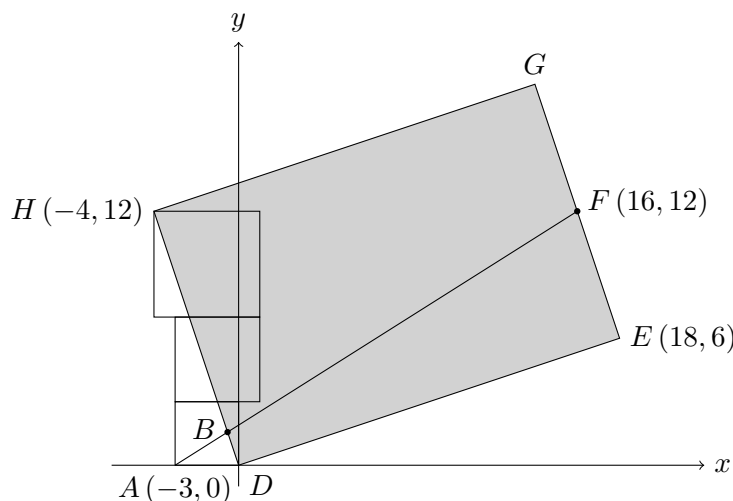
(16 points)

Solution: a) Let T be the point of the top side of the 50 m square directly above D . In the right triangle HTD , one leg is $TD = 50 + 40 + 30 = 120$ m and the other is $HT = 40$ m, so

$$HD^2 = 120^2 + 40^2 = 16\,000, \quad HD = 40\sqrt{10}$$

The other side of the rectangle is one and a half times this, $DE = 60\sqrt{10}$. So the area of the garage is $40\sqrt{10} \cdot 60\sqrt{10} = 24\,000$ m².

- b) Place the figure in a coordinate system with D at the origin and a unit of 10 m on both axes. Then $A(-3, 0)$ and $H(-4, 12)$. We get E by rotating the vector $\overrightarrow{DH}(-4, 12)$ by -90° and multiplying it by 1.5: $E(18, 6)$. Adding $\frac{1}{2}\overrightarrow{DH}$ to $\overrightarrow{DE}$ gives $F(16, 12)$.



The equation of the line DH is $y = -3x$, and the equation of the line AF is $12x - 19y = -36$. Their intersection is $B\left(-\frac{12}{23}, \frac{36}{23}\right)$. So

$$BF = \sqrt{\left(\frac{380}{23}\right)^2 + \left(\frac{240}{23}\right)^2} \approx 19.54,$$

that is, $BF \approx 195.4$ m.

7. Let $f(x) = x^2 + bx + c$, defined on the set of real numbers. For some real number k , $f(k) = 190.1$. Compute the following sum.

$$f(k-1) + f(k+1) + f(k-2) + f(k+2) + \cdots + f(k-5) + f(k+5)$$

(16 points)

Solution: Let us write out the terms $f(k-i) + f(k+i)$ in detail, where $i = 1, 2, 3, 4, 5$.

$$\begin{aligned} f(k-i) + f(k+i) &= (k-i)^2 + b(k-i) + c + (k+i)^2 + b(k+i) + c \\ &= 2k^2 + 2bk + 2c + 2i^2 = 2(k^2 + bk + c) + 2i^2 = 2f(k) + 2i^2 \end{aligned}$$

So the sum is

$$10 \cdot f(k) + 2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) = 10 \cdot 190.1 + 2 \cdot 55 = 1901 + 110 = 2011$$

8. a) Solve the inequality $\sqrt{\log_{\frac{1}{2}}^2 x - 10 \log_{\frac{1}{2}} x + 25} \geq 1$ over the real numbers.

b) Let $0 \leq x \leq 5$. Prove that

$$\sqrt{-x^2 - 2x + 35} + \sqrt{x + 20 - x^2} \leq 10.5$$

(16 points)

Solution: a) Because of the logarithms, the inequality only makes sense for $x > 0$. Under the square root there is a perfect square:

$$\sqrt{\left(\log_{\frac{1}{2}} x - 5\right)^2} \geq 1, \quad \text{that is,} \quad \left|\log_{\frac{1}{2}} x - 5\right| \geq 1$$

So $\log_{\frac{1}{2}} x - 5 \geq 1$ or $\log_{\frac{1}{2}} x - 5 \leq -1$.

In the first case $\log_{\frac{1}{2}} x \geq 6 = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^6$, and since the function $\log_{\frac{1}{2}} x$ is strictly decreasing, $x \leq \left(\frac{1}{2}\right)^6 = \frac{1}{64}$.

In the second case $\log_{\frac{1}{2}} x \leq 4 = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^4$, so $x \geq \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

So the solutions of the inequality are $0 < x \leq \frac{1}{64}$ or $x \geq \frac{1}{16}$.

b) We factor the quadratic expressions under the square roots. The roots of $-x^2 - 2x + 35$ are -7 and 5 , and the roots of $x + 20 - x^2$ are -4 and 5 . So the inequality can be written as

$$\sqrt{(x+7)(5-x)} + \sqrt{(x+4)(5-x)} \leq 10.5$$

Since $0 \leq x \leq 5$, all factors under the square roots are nonnegative. By the inequality between the arithmetic and geometric means,

$$\sqrt{(x+7)(5-x)} \leq \frac{(x+7) + (5-x)}{2} = 6, \quad \text{and} \quad \sqrt{(x+4)(5-x)} \leq \frac{(x+4) + (5-x)}{2} = \frac{9}{2}$$

Adding these two inequalities gives the statement: the sum is at most $6 + \frac{9}{2} = 10.5$.

9. Consider the family of functions defined on the set of real numbers

$$f(x) = x^3 + (a-1)x^2 - (2a+2)x + a - 4$$

- Prove that if a is an integer, then $f(-1) + f(0) + f(1) + f(2)$ is divisible by 6.
- Find the zeros of the function if $a = 4$.
- For some value of a , the tangent line to the graph of the function at $x = 2$ passes through the origin. Find the equation of this tangent line.

(16 points)

Solution: a) $f(-1) = -1 + a - 1 + 2a + 2 + a - 4 = 4a - 4$, $f(0) = a - 4$, $f(1) = 1 + a - 1 - 2a - 2 + a - 4 = -6$, and $f(2) = 8 + 4a - 4 - 4a - 4 + a - 4 = a - 4$. So

$$f(-1) + f(0) + f(1) + f(2) = (4a - 4) + (a - 4) - 6 + (a - 4) = 6a - 18 = 6(a - 3)$$

which is divisible by 6 if a is an integer.

b) If $a = 4$, then $f(x) = x^3 + 3x^2 - 10x = x(x^2 + 3x - 10) = x(x - 2)(x + 5)$. So the zeros of the function are 0, 2, and -5.

c) $f(2) = a - 4$, so the point of tangency is $(2, a - 4)$. The slope of the tangent line is the derivative of the function at $x = 2$:

$$f'(x) = 3x^2 + 2(a-1)x - 2a - 2, \quad \text{so} \quad f'(2) = 12 + 4a - 4 - 2a - 2 = 2a + 6$$

So the equation of the tangent line is $y - (a - 4) = (2a + 6)(x - 2)$. It passes through the origin, so $-a + 4 = -4a - 12$, and $a = -\frac{16}{3}$. Then the slope is $2a + 6 = -\frac{14}{3}$, and the tangent line is

$$y = -\frac{14}{3}x$$