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translated from: KoMaL, October 2011, pages 396-397

Pdf file: KoMaL, October 2011, pages 396-397

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the real solutions of the following equations.

a) $(x + 500)^2 = 252\,012 - 1011x$

b) $\lg^2 x + 2011 \cdot \lg x - 2012 = 0$

c) $x + 2011\sqrt{x-7} - 2019 = 0$

(11 points)

Solution: a) Expanding the square and rearranging, we get $x^2 + 2011x - 2012 = 0$. By the relationship between the roots and the coefficients ($x_1 + x_2 = -2011$, $x_1x_2 = -2012$), or by the quadratic formula, $x_1 = 1$ and $x_2 = -2012$.

b) The equation is quadratic in $\lg x$, and its domain is $x > 0$. By part a), $\lg x = 1$ or $\lg x = -2012$, so $x_1 = 10$ and $x_2 = 10^{-2012}$. Both are solutions, since both are in the domain.

c) The domain is $x \geq 7$. The equation can be written as $(x - 7) + 2011\sqrt{x-7} - 2012 = 0$, a quadratic equation in $\sqrt{x-7}$. So $\sqrt{x-7} = 1$ or $\sqrt{x-7} = -2012$. The first case gives $x = 8$; the second has no solution. So the only solution is $x = 8$.

2. The first term of a geometric sequence is 2. If we decrease the second term of the sequence by 20, and write the third term in front of the first two, we get three consecutive terms of an arithmetic sequence. Find the original three numbers.

(12 points)

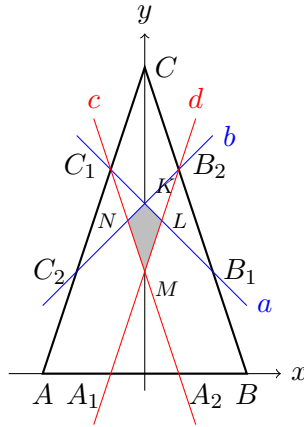
Solution: Let the common ratio of the geometric sequence be q ; then its first three terms are 2, $2q$, $2q^2$. By the conditions, $2q^2$, 2, $2q - 20$ are three consecutive terms of an arithmetic sequence, so $2 - 2q^2 = (2q - 20) - 2$. Rearranging, $q^2 + q - 12 = 0$, so $q_1 = 3$ and $q_2 = -4$.

There are two solutions: 2, 6, 18 and 2, -8, 32.

3. In the isosceles triangle ABC , $AB = 6$ and $CF = 9$, where F is the midpoint of the base AB . Starting from the vertex A and going around the triangle in the direction A, B, C , the points dividing the sides into three equal parts are $A_1, A_2, B_1, B_2, C_1, C_2$. Find the area of the quadrilateral formed by the lines C_1B_1, C_2B_2, C_1A_2 , and A_1B_2 .

(14 points)

Solution: Place the triangle in a coordinate system: $A(-3, 0), B(3, 0), C(0, 9)$. Then the points dividing the sides into three equal parts are $A_1(-1, 0), A_2(1, 0), B_1(2, 3), B_2(1, 6), C_1(-1, 6), C_2(-2, 3)$.



The equations of the lines through two given points:

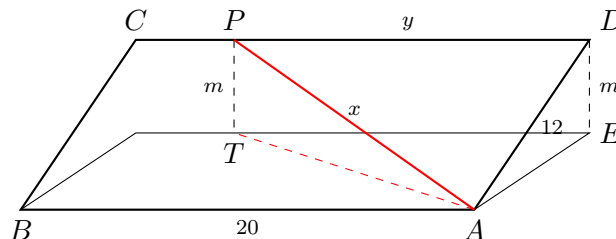
$$a = C_1B_1: y = -x + 5, \quad b = C_2B_2: y = x + 5, \quad c = C_1A_2: y = -3x + 3, \quad d = A_1B_2: y = 3x + 3$$

The intersection points are: a and b : $K(0, 5)$; c and d : $M(0, 3)$; a and d : $L(\frac{1}{2}, \frac{9}{2})$; and by symmetry, c and b : $N(-\frac{1}{2}, \frac{9}{2})$. By symmetry, $KLMN$ is a kite, with diagonals $KM = 2$ and $NL = 1$. So its area is $\frac{2 \cdot 1}{2} = 1$.

4. The lower side AB of a rectangle $ABCD$ inclined at 20° is horizontal and 20 meters long. We want to get from the vertex A along a straight line to some point of the side CD . The parallel sides AB and CD are 12 meters apart. Where can we arrive, if the angle between the path and the horizontal may not exceed 12° ?

(14 points)

Solution: Let P be the point of CD where we arrive, let E be the foot of the perpendicular from D to the horizontal plane through AB , and let $m = DE$ be the height of the side CD above that plane.



In the right triangle AED , $\sin 20^\circ = \frac{m}{12}$, so $m = 12 \sin 20^\circ \approx 4.104$. If T is the foot of the perpendicular from P , then $PT = m$, and the angle of the path AP with the horizontal is $\angle PAT$. If this angle is exactly 12° , then in the right triangle ATP , $\sin 12^\circ = \frac{m}{x}$, where $x = AP$, so $x = \frac{m}{\sin 12^\circ} \approx 19.740$. Finally, in the right triangle ADP ,

$$y = DP = \sqrt{x^2 - 12^2} \approx \sqrt{19.740^2 - 144} \approx 15.67$$

The farther P is from D , the longer the path and the smaller its angle with the horizontal. So the angle is at most 12° exactly when $DP \geq 15.67$ m, that is, when P is on CD at most about 4.33 m from C .

Note: KoMaL rounds m to 4.1 before dividing and gets $DP \geq 15.65$ m; without the early rounding, the bound is 15.67 m.

Part II

5. Consider the circle k with center $K(5, 2)$ and radius $r = 3$. The tangent lines e_1 and e_2 can be drawn from the origin to the circle k .
- Write the equations of e_1 and e_2 .
 - What is the angle between e_1 and e_2 ?
 - Find the area of the plane region from whose points the circle k is seen at an angle of at least 60° and at most 90° .

(16 points)

Solution: a) The equation of the circle k is $(x - 5)^2 + (y - 2)^2 = 9$, and the lines through the origin (other than the y -axis, which is not tangent) are $y = mx$. Such a line is tangent to the circle if it has exactly one common point with the circle, that is, if the equation

$$(x - 5)^2 + (mx - 2)^2 = 9, \quad \text{i.e.} \quad (m^2 + 1)x^2 - (4m + 10)x + 20 = 0$$

has exactly one solution. So its discriminant is zero:

$$(4m + 10)^2 - 80(m^2 + 1) = 0, \quad 16m^2 - 20m - 5 = 0, \quad m_{1,2} = \frac{20 \pm \sqrt{720}}{32} = \frac{5 \pm 3\sqrt{5}}{8}$$

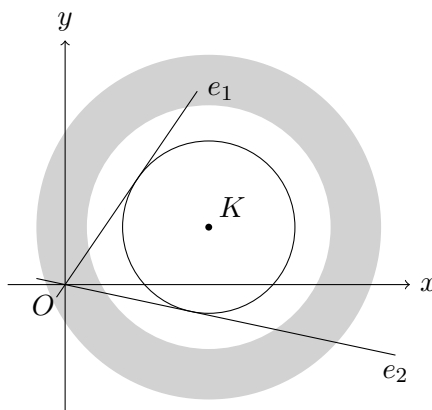
So the tangent lines are $e_1: y = \frac{5+3\sqrt{5}}{8}x$ and $e_2: y = \frac{5-3\sqrt{5}}{8}x$.

b) Let E_1 be the point of tangency on e_1 . The angle between the two tangent lines is twice the acute angle φ at O of the right triangle $OK E_1$. Here $OK = \sqrt{5^2 + 2^2} = \sqrt{29}$ and $E_1 K = 3$, so $\sin \varphi = \frac{3}{\sqrt{29}}$. So $\varphi \approx 33.85^\circ$, and the angle between the tangent lines is $2\varphi \approx 67.7^\circ$.

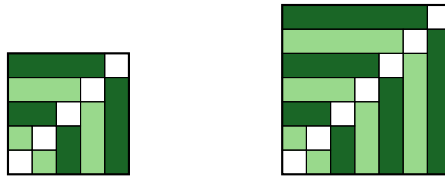
c) If the circle is seen from a point P at an angle of 60° , let R_1 be the point of tangency of one of the tangent lines from P . In the right triangle $K P R_1$, $\angle K P R_1 = 30^\circ$ and $K R_1 = 3$, so $K P = 6$. So these points form the circle with center K and radius 6.

Similarly, if the circle is seen from Q at an angle of 90° , with point of tangency T_1 , then in the right triangle $K Q T_1$, $\angle K Q T_1 = 45^\circ$, so $K Q = 3\sqrt{2}$. These points form the circle with center K and radius $3\sqrt{2}$.

The closer a point is to K , the larger the angle, so the region is the ring between these two circles (shaded below). Its area is $(36 - 18)\pi = 18\pi \approx 56.5$.



6. The waiting room of the Tiszai railway station in Miskolc was paved partly with the tiles shown in the figure. On these tiles a 5×5 square grid was colored with three colors (dark green, light green, and white). This pattern can be continued to the right and upward, and the coloring can be made for any odd number greater than five. The figure also shows the 7×7 grid made this way.



- What fraction of the square with side 51 units is light green?
- What fraction of the square with side 101 units is dark green?
- Prove that if the side length of the square increases without bound, then the area of both the light green and the dark green parts approaches half of the area of the square.

(16 points)

Solution: a) The square with side 51 has area $51^2 = 2601$. The areas of the light green rectangles are 1, 3, 5, ..., 49, and there are two of each, so their total area is $2(1 + 3 + 5 + \dots + 49) = 2 \cdot 25^2 = 1250$. (The sum of the first 25 positive odd numbers is 25^2 .) So $\frac{1250}{2601} \approx 0.4806$ of the square is light green.

b) The square with side 101 has area $101^2 = 10201$. The areas of the dark green rectangles are 2, 4, 6, ..., 100, and there are two of each, so their total area is

$$2(2 + 4 + 6 + \dots + 100) = 4(1 + 2 + 3 + \dots + 50) = 4 \cdot \frac{50 \cdot 51}{2} = 5100$$

So $\frac{5100}{10201} \approx 0.49995$ of the square is dark green.

c) The square with side $2n + 1$ (where $n > 1$ is an integer) has area $(2n + 1)^2$. The light green rectangles have areas 1, 3, ..., $2n - 1$, two of each, with total area $2(1 + 3 + \dots + (2n - 1)) = 2n^2$; the dark green rectangles have areas 2, 4, ..., $2n$, two of each, with total area $4(1 + 2 + \dots + n) = 2n(n + 1)$. So

$$\lim_{n \rightarrow \infty} \frac{2n^2}{(2n + 1)^2} = \lim_{n \rightarrow \infty} \frac{2}{4 + \frac{4}{n} + \frac{1}{n^2}} = \frac{1}{2}, \quad \lim_{n \rightarrow \infty} \frac{2n(n + 1)}{(2n + 1)^2} = \lim_{n \rightarrow \infty} \frac{2 + \frac{2}{n}}{4 + \frac{4}{n} + \frac{1}{n^2}} = \frac{1}{2}$$

which proves the statement in both cases.

7. The graph of the function $f(x) = \frac{x^2}{16} + 1$ defined on $[0, 4]$ is rotated about the x -axis. The surface obtained is the lateral surface of a vessel in the shape of a solid of revolution, whose bottom is the smaller circle. The unit of the coordinate system is 1 dm.

- How many liters does the vessel hold, if the thickness of its wall is negligible?
- A straight rod of negligible thickness stands at the center of the bottom of the vessel. We tilt the rod so that its lower end does not move. At what height does the rod touch the wall of the vessel?

(16 points)

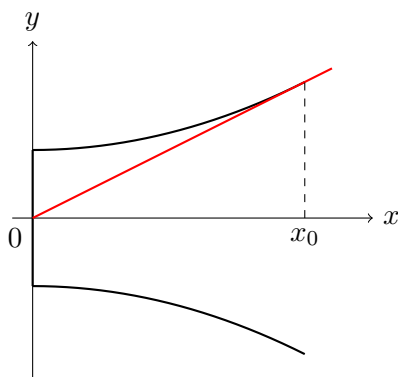
Solution: a) The volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^4 \left(\frac{x^2}{16} + 1 \right)^2 dx = \pi \int_0^4 \left(\frac{x^4}{256} + \frac{x^2}{8} + 1 \right) dx \\ &= \pi \left[\frac{x^5}{1280} + \frac{x^3}{24} + x \right]_0^4 = \pi \left(\frac{4}{5} + \frac{8}{3} + 4 \right) = \frac{112\pi}{15} \approx 23.5 \end{aligned}$$

So the vessel holds about 23.5 liters.

Note: KoMaL writes $\frac{52\pi}{15}$, a misprint; its value 23.5 is correct for $\frac{112\pi}{15}$.

b) The figure shows the cross-section through the axis of rotation, with the tilted rod. The height in question is x_0 .



The rod lies on a line $y = mx$ through the origin (with $m > 0$). It touches the wall where this line is tangent to the curve $y = \frac{x^2}{16} + 1$, that is, where the equation $mx = \frac{x^2}{16} + 1$, i.e. $x^2 - 16mx + 16 = 0$, has exactly one solution: $256m^2 - 64 = 0$, so $m = \frac{1}{2}$. The tangent line $y = \frac{x}{2}$ touches the curve at the point $(4, 2)$.

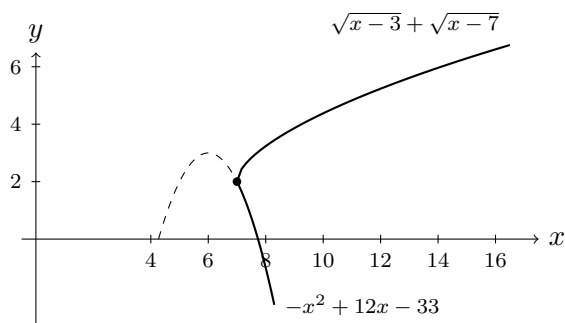
So $x_0 = 4$: the rod touches the vessel at a height of 4 dm, at its upper rim.

8. Solve the equation $-x^2 + 12x - 33 = \sqrt{x-3} + \sqrt{x-7}$.

(16 points)

Solution: The domain of the equation is $x \geq 7$. We rewrite the quadratic expression on the left side:

$$-x^2 + 12x - 33 = -(x^2 - 12x + 33) = -(x-6)^2 + 3$$



In this form it is easy to see that this expression has its maximum at $x = 6$ and is strictly decreasing on $[6, \infty)$, so also on the domain of the equation.

Since $\sqrt{x-3}$ and $\sqrt{x-7}$ are strictly increasing, so is their sum. For the smallest element of the domain, $x = 7$, the left side is $-49 + 84 - 33 = 2$, and the right side is $\sqrt{4} + \sqrt{0} = 2$. So $x = 7$ is a solution of the equation, and by the monotonicity there are no other solutions.

9. We know the lengths of all three sides and the measures of all three angles of a triangle. Show that substituting these values into the formula

$$\frac{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2}{\cot \alpha + \cot \beta + \cot \gamma}$$

gives the area of the triangle.

(16 points)

Solution: Let t be the area of the triangle. We rewrite the formula:

$$\frac{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2}{\cot \alpha + \cot \beta + \cot \gamma} = \frac{a^2 + b^2 + c^2}{4 \left(\frac{\cos \alpha}{\sin \alpha} + \frac{\cos \beta}{\sin \beta} + \frac{\cos \gamma}{\sin \gamma} \right)}$$

By the law of cosines, $\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}$, and since $t = \frac{bc \sin \alpha}{2}$, we have $2bc \sin \alpha = 4t$. So

$$\frac{\cos \alpha}{\sin \alpha} = \frac{b^2 + c^2 - a^2}{2bc \sin \alpha} = \frac{b^2 + c^2 - a^2}{4t}$$

In the same way, $\frac{\cos \beta}{\sin \beta} = \frac{a^2 + c^2 - b^2}{4t}$ and $\frac{\cos \gamma}{\sin \gamma} = \frac{a^2 + b^2 - c^2}{4t}$. Adding these three, the numerator is $(b^2 + c^2 - a^2) + (a^2 + c^2 - b^2) + (a^2 + b^2 - c^2) = a^2 + b^2 + c^2$, so

$$\frac{a^2 + b^2 + c^2}{4 \cdot \frac{a^2 + b^2 + c^2}{4t}} = t$$

So the formula indeed gives the area of the triangle.