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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

## Part I

1. The sides of a rectangle are in the ratio 1 : 2. We know that the measure of its area equals the measure of the sum of its perimeter and the lengths of its diagonals. Find the distance of a vertex of the rectangle from the diagonal not containing that vertex.

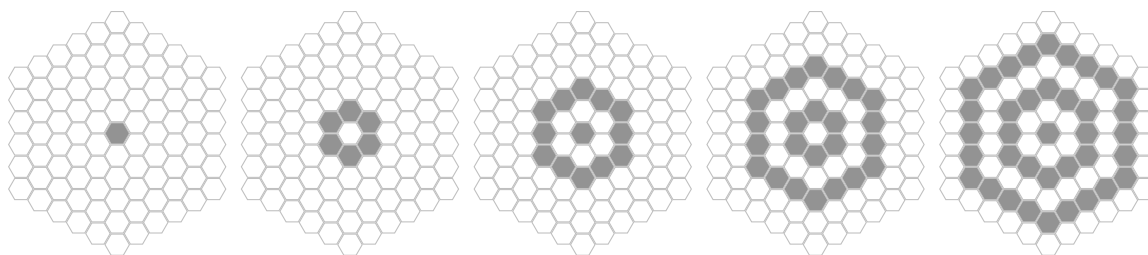
(11 points)

2. Solve the following system of equations.

$$\begin{aligned}x^2 - y^2 - 8x + 8y &= 0 \\ xy - 3x - y &= 0\end{aligned}$$

(12 points)

3. Paving stones of many shapes and colors are available for paving squares. The figures show patterns formed from hexagonal paving stones: in each pattern some rings of stones are dark gray.



- a) How many dark gray stones are needed for the sixth pattern?  
b) For which pattern are exactly 150 dark gray stones needed?  
c) Give a recursive formula for the number of dark gray stones in the  $n$ th pattern.
4. At a school chess tournament everyone played everyone else exactly once. After 63 games, everyone still had four games left.
- a) How many players were in the tournament?  
b) At that time an editor of the school paper talked with two randomly chosen players. What is the probability that they had not yet played each other?

(14 points)

(14 points)

## Part II

5. a) If we decrease the roots of the equation  $ax^2 + bx + c = 0$  by 2, we get the roots of the equation  $ax^2 + cx + b = 0$ . Find the coefficients of the original equation, if their sum is  $-3$ .
- b) In the equation  $ax^2 + bx + c = 0$  the coefficients are three consecutive terms of an increasing arithmetic sequence, and in the equation  $(a + 2)x^2 + bx + (c + 2) = 0$  the coefficients are three consecutive terms of a geometric sequence. Does the first equation have a real solution, if the sum of its coefficients is 9?
- c) The quadratic equation  $ax^2 + bx + c = 0$  has two zeros,  $x_1$  and  $x_2$ . Write, in factored form, a cubic equation whose zeros are  $x_1x_2$ ,  $x_1^2 + x_2^2$ , and  $\frac{1}{x_1} + \frac{1}{x_2}$ .

(16 points)

6. The circle  $x^2 + y^2 = 25$ , the graph of the function

$$f(x) = \frac{|x + 2| + |x - 2| - 2}{2}$$

and the  $x$ -axis determine a plane region, which we rotate about the  $x$ -axis. What is the volume of the resulting solid of revolution?

(16 points)

7. a) Prove that  $\sqrt{31 + 8\sqrt{15}} + \sqrt{31 - 8\sqrt{15}}$  is an integer.
- b) Solve the following equation over the real numbers.

$$\sqrt{\left(31 + 8\sqrt{15}\right)^x} + \sqrt{\left(31 - 8\sqrt{15}\right)^x} = \frac{10}{3}$$

(16 points)

8. At the back of an open-air stage stands a column 12 m high, whose top is seen from the prompter's box at an angle of elevation of  $16^\circ$ . Above the line joining the foot of the column and the prompter's box there is a platform, from which a 180 cm tall actor sees the top of the column at an angle of elevation of  $22^\circ$ . From the prompter's box the top of the actor's head is seen at an angle of elevation of  $10^\circ$ .
- a) Prove that the top of the actor's head is the same distance from the top of the column and from the prompter's box.
- b) How high is the platform built on the stage?

(16 points)

9. a) 100 red and green building blocks were put in a box, some small and some large. Drawing a small red block at random is as likely as drawing a large red or a small green one. The ratio of the green blocks to the small blocks is  $7 : 11$ . There are 20 fewer large blocks than red ones. How many blocks of each kind are there?
- b) 93 red and green building blocks were put in a box, some small and some large. The number of blocks of each kind is a different prime. The number of red blocks is divisible by seven. There are fewest small green blocks. There are fifty more large red blocks than small red ones. How many blocks of each kind are there?

(16 points)