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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The sides of a rectangle are in the ratio 1 : 2. We know that the measure of its area equals the measure of the sum of its perimeter and the lengths of its diagonals. Find the distance of a vertex of the rectangle from the diagonal not containing that vertex.

(11 points)

Solution: Let the sides of the rectangle be a and $2a$; then its diagonal is $a\sqrt{5}$. The area of the rectangle is $2a^2$, and the sum of its perimeter and the lengths of its diagonals is $6a + 2a\sqrt{5} = 2a(3 + \sqrt{5})$. So $2a^2 = 2a(3 + \sqrt{5})$, and since $a \neq 0$, $a = 3 + \sqrt{5}$.

The area of the rectangle can also be computed from a diagonal and the distance m of a vertex from it: $2 \cdot \frac{a\sqrt{5} \cdot m}{2} = 2a^2$, so

$$m = \frac{2a}{\sqrt{5}} = \frac{2(3 + \sqrt{5})}{\sqrt{5}} = \frac{6\sqrt{5} + 10}{5} \approx 4.68$$

2. Solve the following system of equations.

$$\begin{aligned} x^2 - y^2 - 8x + 8y &= 0 \\ xy - 3x - y &= 0 \end{aligned}$$

(12 points)

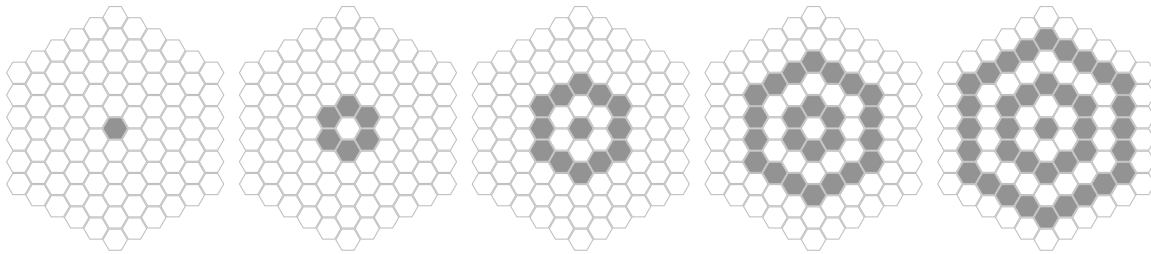
Solution: Factoring the first equation: $(x - y)(x + y - 8) = 0$, so $y = x$ or $y = -x + 8$. In both cases, substituting into the second equation gives a quadratic equation in x .

If $y = x$: $x^2 - 4x = 0$, so $(0, 0)$ and $(4, 4)$.

If $y = -x + 8$: $x^2 - 6x + 8 = 0$, so $(2, 6)$ and $(4, 4)$; the second one was already found.

So the system has three solutions: $(0, 0)$, $(4, 4)$, and $(2, 6)$.

3. Paving stones of many shapes and colors are available for paving squares. The figures show patterns formed from hexagonal paving stones: in each pattern some rings of stones are dark gray.



- How many dark gray stones are needed for the sixth pattern?
- For which pattern are exactly 150 dark gray stones needed?
- Give a recursive formula for the number of dark gray stones in the n th pattern.

(14 points)

Solution: a) Studying the patterns, the sixth pattern can be obtained from the fourth: after a light ring is laid around it, 6 dark gray stones must be placed along each of the six sides, but then the corners are counted twice. Since the fourth pattern contains 24 dark gray stones, the sixth needs $24 + 6 \cdot 6 - 6 = 54$ dark gray stones.

b) Using the same idea, the numbers of dark gray stones in consecutive patterns are 1, 6, 13, 24, 37, 54, 73, 96, 121, 150, ... So the 10th pattern needs exactly 150 dark gray stones.

c) $a_1 = 1$, $a_2 = 6$, and for $n > 2$: $a_n = a_{n-2} + 6n - 6$.

4. At a school chess tournament everyone played everyone else exactly once. After 63 games, everyone still had four games left.

- How many players were in the tournament?
- At that time an editor of the school paper talked with two randomly chosen players. What is the probability that they had not yet played each other?

(14 points)

Solution: a) Let the number of players be n ; the total number of games is $\frac{n(n-1)}{2}$. Everyone still had four games left, so $\frac{4n}{2}$ games were still to be played, and the number of games played was

$$\frac{n(n-1)}{2} - \frac{4n}{2} = 63$$

that is, $n^2 - 5n - 126 = 0$, with roots 14 and -9 . So 14 players took part in the tournament.

b) The total number of games is $\frac{14 \cdot 13}{2} = 91$, and $91 - 63 = 28$ of them have not been played, so the probability is $\frac{28}{91} = \frac{4}{13} \approx 0.31$.

Part II

5. a) If we decrease the roots of the equation $ax^2 + bx + c = 0$ by 2, we get the roots of the equation $ax^2 + cx + b = 0$. Find the coefficients of the original equation, if their sum is -3 .
- b) In the equation $ax^2 + bx + c = 0$ the coefficients are three consecutive terms of an increasing arithmetic sequence, and in the equation $(a + 2)x^2 + bx + (c + 2) = 0$ the coefficients are three consecutive terms of a geometric sequence. Does the first equation have a real solution, if the sum of its coefficients is 9?
- c) The quadratic equation $ax^2 + bx + c = 0$ has two zeros, x_1 and x_2 . Write, in factored form, a cubic equation whose zeros are x_1x_2 , $x_1^2 + x_2^2$, and $\frac{1}{x_1} + \frac{1}{x_2}$.

(16 points)

Solution: a) Let the roots of the second equation be x_1 and x_2 ; then $x_1 + x_2 = -\frac{c}{a}$ and $x_1x_2 = \frac{b}{a}$. The roots of the first equation are $x_1 + 2$ and $x_2 + 2$, so

$$\begin{aligned}x_1 + 2 + x_2 + 2 &= -\frac{c}{a} + 4 = -\frac{b}{a} \\(x_1 + 2)(x_2 + 2) &= \frac{b}{a} + 2\left(-\frac{c}{a}\right) + 4 = \frac{c}{a}\end{aligned}$$

Together with $a + b + c = -3$, we get the system $a + b + c = -3$, $4a + b - c = 0$, $4a + b - 3c = 0$, whose solution is $a = 1$, $b = -4$, $c = 0$. (Indeed, $x^2 - 4x = 0$ has roots 0 and 4, and $x^2 - 4 = 0$ has roots -2 and 2 .)

b) By the conditions, $2b = a + c$, $b^2 = (a + 2)(c + 2)$, and $a + b + c = 9$. From the first and third equations, $b = 3$, and then $a + c = 6$ and $(a + 2)(c + 2) = 9$. Substituting $c = 6 - a$: $a^2 - 6a - 7 = 0$. Only the solution less than $b = 3$ is acceptable, since the coefficients form an increasing arithmetic sequence: $a = -1$, and $c = 7$. The discriminant of the first equation is $D = 3^2 - 4 \cdot (-1) \cdot 7 > 0$, so it has real solutions.

c) Since $x_1x_2 = \frac{c}{a}$ and $x_1 + x_2 = -\frac{b}{a}$,

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = \frac{b^2 - 2ac}{a^2}, \quad \frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1x_2} = -\frac{b}{c}$$

So a suitable cubic equation is

$$\left(x - \frac{c}{a}\right) \left(x - \frac{b^2 - 2ac}{a^2}\right) \left(x + \frac{b}{c}\right) = 0$$

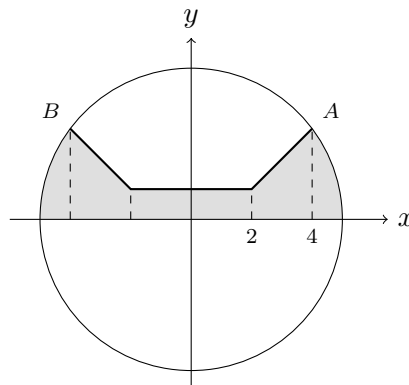
6. The circle $x^2 + y^2 = 25$, the graph of the function

$$f(x) = \frac{|x + 2| + |x - 2| - 2}{2}$$

and the x -axis determine a plane region, which we rotate about the x -axis. What is the volume of the resulting solid of revolution?

(16 points)

Solution: Breaking f into pieces: $f(x) = -x - 1$ if $x \leq -2$; $f(x) = 1$ if $-2 \leq x \leq 2$; $f(x) = x - 1$ if $x \geq 2$. The graph meets the circle at $A(4, 3)$ and $B(-4, 3)$. The region is bounded below by the x -axis and above by the graph of f for $-4 \leq x \leq 4$ and by the circle for $4 \leq |x| \leq 5$.



By symmetry the volume is twice the volume over $[0, 5]$:

$$V = 2\pi \left[\int_0^2 1 \, dx + \int_2^4 (x-1)^2 \, dx + \int_4^5 (25-x^2) \, dx \right] = 2\pi \left(2 + \frac{26}{3} + \frac{14}{3} \right) = \frac{92}{3}\pi$$

7. a) Prove that $\sqrt{31+8\sqrt{15}} + \sqrt{31-8\sqrt{15}}$ is an integer.
 b) Solve the following equation over the real numbers.

$$\sqrt{(31+8\sqrt{15})^x} + \sqrt{(31-8\sqrt{15})^x} = \frac{10}{3}$$

(16 points)

Solution: a) Since $31 \pm 8\sqrt{15} = (4 \pm \sqrt{15})^2$,

$$\sqrt{31+8\sqrt{15}} + \sqrt{31-8\sqrt{15}} = 4 + \sqrt{15} + 4 - \sqrt{15} = 8$$

which is indeed an integer.

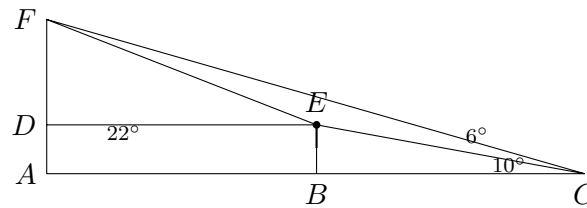
b) The equation is $(4 + \sqrt{15})^x + (4 - \sqrt{15})^x = \frac{10}{3}$. Let $(4 + \sqrt{15})^x = a$; then $(4 - \sqrt{15})^x = \frac{1}{a}$, since $(4 + \sqrt{15})(4 - \sqrt{15}) = 1$. From $a + \frac{1}{a} = \frac{10}{3}$, $3a^2 - 10a + 3 = 0$, so $a = 3$ or $a = \frac{1}{3}$, and

$$x_1 = \log_{4+\sqrt{15}} 3 \approx 0.532 \quad \text{and} \quad x_2 = \log_{4+\sqrt{15}} \frac{1}{3} = -\log_{4+\sqrt{15}} 3 \approx -0.532$$

8. At the back of an open-air stage stands a column 12 m high, whose top is seen from the prompter's box at an angle of elevation of 16° . Above the line joining the foot of the column and the prompter's box there is a platform, from which a 180 cm tall actor sees the top of the column at an angle of elevation of 22° . From the prompter's box the top of the actor's head is seen at an angle of elevation of 10° .
- a) Prove that the top of the actor's head is the same distance from the top of the column and from the prompter's box.
 b) How high is the platform built on the stage?

(16 points)

Solution: Let A be the foot and F the top of the column, C the prompter's box, E the top of the actor's head, B the foot of the platform, and D the point of the column at the height of E .



a) We know that $AF = 12$ m, $\angle ACE = 10^\circ$, $\angle ACF = 16^\circ$, and $\angle DEF = 22^\circ$. So $\angle ECF = \angle ACF - \angle ACE = 6^\circ$, and $\angle CFE = \angle CFA - \angle EFD = 74^\circ - 68^\circ = 6^\circ$. So the triangle EFC is isosceles, and indeed $EC = EF$.

b) In the right triangle ACF , $CF = \frac{12}{\sin 16^\circ}$. Since EFC is isosceles, the foot G of its altitude from E is the midpoint of CF , so $GC = \frac{6}{\sin 16^\circ}$. In the right triangle EGC , $EC = \frac{GC}{\cos 6^\circ}$, and in the right triangle EBC ,

$$EB = EC \sin 10^\circ = \frac{6 \sin 10^\circ}{\cos 6^\circ \sin 16^\circ} \approx 3.8$$

Since the actor is 1.8 m tall, the platform is about 2 m high.

9. a) 100 red and green building blocks were put in a box, some small and some large. Drawing a small red block at random is as likely as drawing a large red or a small green one. The ratio of the green blocks to the small blocks is 7 : 11. There are 20 fewer large blocks than red ones. How many blocks of each kind are there?
- b) 93 red and green building blocks were put in a box, some small and some large. The number of blocks of each kind is a different prime. The number of red blocks is divisible by seven. There are fewest small green blocks. There are fifty more large red blocks than small red ones. How many blocks of each kind are there?

(16 points)

Solution: a) Let the numbers of large red, small red, large green, and small green blocks be a, b, c, d . By the conditions,

$$a + b + c + d = 100, \quad a + d = b, \quad \frac{c + d}{b + d} = \frac{7}{11}, \quad a + c = a + b - 20$$

The solution of the system is $a = 25$ (large red), $b = 40$ (small red), $c = 20$ (large green), $d = 15$ (small green).

b) The sum of four different odd primes would be even, so one of the numbers is 2; by the conditions, this is the number of small green blocks. So the red blocks and the large green blocks together number 91. The number of red blocks is divisible by seven, so the number of large green blocks is also divisible by seven; the only such prime is 7. So there are 7 large green blocks and 84 red ones. There are fifty more large red blocks than small red ones, so there are 67 large red and 17 small red blocks. (All four numbers, 67, 17, 7, 2, are primes.)