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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The common difference of an arithmetic sequence is 56, and its n th term is 2012. The sum of the first n terms of the sequence is 32 000. Find the first term of the sequence.

(12 points)

Solution: The n th term of the sequence is $2012 = a_1 + (n - 1) \cdot 56$, so $a_1 = 2068 - 56n$. The sum of the first n terms is

$$32\,000 = \frac{[2a_1 + (n - 1) \cdot 56] n}{2} = \frac{[2(2068 - 56n) + (n - 1) \cdot 56] n}{2} = \frac{(4080 - 56n) n}{2} = (2040 - 28n) n$$

Rearranging and dividing by 4: $7n^2 - 510n + 8000 = 0$. One root of this quadratic equation, $\frac{160}{7}$, is not an integer, so it is not a solution of the problem; the other root is $n = 50$. So the first term of the sequence is

$$a_1 = 2068 - 56 \cdot 50 = -732$$

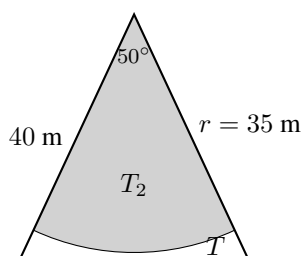
Substituting back, we can check that this is indeed a solution.

2. A city park has the shape of an isosceles triangle whose legs are 40 meters long and form an angle of 50° . An arc of radius 35 meters is drawn about the meeting point of the legs. The part inside the arc is sown with grass, and the part outside the arc is planted with flowers.

- a) Find the number of flower plants needed for the planting, if they count on 30 plants per square meter.
b) Paths are planned all around the boundary of the park and along the part of the arc inside the park. Find the total length of the paths to be built. (Neglect the width of the paths in the calculations.)

(13 points)

Solution: a) The whole arc lies inside the park, since the altitude of the isosceles triangle to its base is $40 \cos 25^\circ \approx 36.25 \text{ m} > 35 \text{ m}$.



The area to be planted with flowers is the difference of the area T_1 of the isosceles triangle and the area T_2 of the circular sector bounded by the arc:

$$\begin{aligned} T_1 &= \frac{40 \cdot 40 \cdot \sin 50^\circ}{2} \approx 612.84 \text{ m}^2 \\ T_2 &= \frac{50^\circ}{360^\circ} \cdot 35^2 \pi \approx 534.51 \text{ m}^2 \\ T &= T_1 - T_2 \approx 78.33 \text{ m}^2 \end{aligned}$$

The number of flower plants needed is 30 times the area in square meters, about 2350.

b) The base of the isosceles triangle is $2 \cdot 40 \cdot \sin 25^\circ \approx 33.81$ m, and the length of the arc is $\frac{50^\circ}{360^\circ} \cdot 2 \cdot 35\pi \approx 30.54$ m. So the total length of the paths is about $40 + 40 + 33.81 + 30.54 \approx 144$ m.

3. How many non-isomorphic simple graphs are there whose degree sequence is

- a) 0, 1, 2, 3, 4, 5, 6, 7; b) 2, 2, 3, 3, 3, 4, 5, 5; c) 2, 3, 3, 4, 5, 5, 6;
d) 3, 3, 3, 5, 6, 6, 6; e) 2, 3, 3, 4, 4, 4?

In the cases where no such graph exists, explain why not.

(14 points)

Solution: a) There is no such graph, since every simple graph has two vertices of the same degree. (Here: the vertex of degree 7 would be joined to all other vertices, including the one of degree 0.)

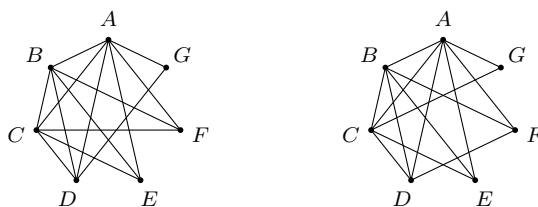
b) There is no such graph, since the sum of the degrees of the vertices of any graph is even, and here it is 27.

c) Let A be the vertex of degree 6, B and C the vertices of degree 5, D the vertex of degree 4, E and F the vertices of degree 3, and finally G the vertex of degree 2.

The vertex A is certainly joined to all the others. If B and C were not joined to each other, then both would be joined to G , and then three edges would start from G ; this is impossible, so the two vertices of degree 5 are joined to each other.

If one of B and C , say B , were not joined to D , then it would be joined to the remaining three vertices E , F , and G . Then edges would have to go from D to C , E , and F as well, but two more edges would still be missing at C , so this is not possible either. So B and C are both joined to D .

Then 5 more edges must start from B , C , and D to E , F , and G ; this can be done in two ways, depending on whether the other endpoint of the one remaining edge of D is a vertex of degree 3 or the vertex of degree 2.

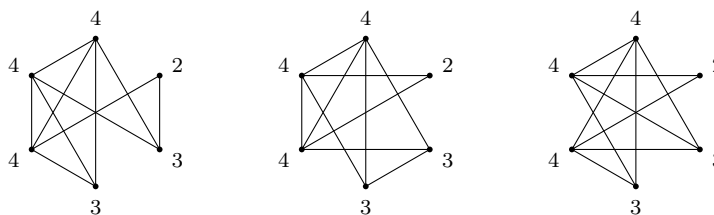


So there are two non-isomorphic simple graphs satisfying the conditions.

d) There is no such graph. The three vertices of degree 6 are joined to all the others, but then no more edges can start from the vertices of degree 3. So two more edges would have to start from the vertex of degree 5, but simple graphs have no loops.

e) If all three possible edges are drawn between the vertices of degree 4, then 6 more edges must go from them to the other three vertices, whose degrees add up to 8. This means that there is one more edge among the other three vertices. So we get two different graphs, depending on whether this edge joins the two vertices of degree 3 or not.

If two of the possible edges are drawn between the vertices of degree 4 and one is not, then 8 edges must go from them to the other three vertices, whose degrees add up to exactly 8, so there are no further edges in this case. The graph is uniquely determined, since the two vertices of degree 4 that are not joined to each other are joined to all three other vertices, and the third vertex of degree 4 only to the two vertices of degree 3.



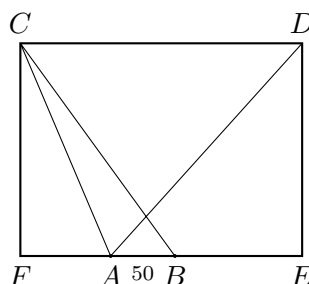
So altogether there are three non-isomorphic simple graphs satisfying the conditions.

(Further edges between the vertices of degree 4 cannot be missing: with at most one edge among them, at least 10 edges would have to go to the other three vertices, whose degrees add up to only 8.)

4. We want to survey a rectangular plot of land. For this we choose points A and B , 50 meters apart, in the interior of one side of the rectangle. Let C and D be the vertices of the rectangle on the side opposite to the side containing the two points. We measure the following angles: $\angle CAB = 113^\circ$, $\angle DAB = 48^\circ$, $\angle CBA = 54^\circ$. What is the area of the plot of land?

(12 points)

Solution: Let E and F be the endpoints of the side of the rectangle containing AB , as in the figure.



By the law of sines in the triangle ABC , the side AC is

$$AC = AB \cdot \frac{\sin 54^\circ}{\sin (180^\circ - 54^\circ - 113^\circ)} = 50 \cdot \frac{\sin 54^\circ}{\sin 13^\circ} \approx 179.81 \text{ m}$$

Using this in the right triangle ACF (where $\angle CAF = 180^\circ - 113^\circ = 67^\circ$):

$$CF = AC \cdot \sin 67^\circ \approx 165.53 \text{ m}, \quad FA = AC \cdot \cos 67^\circ \approx 70.26 \text{ m}$$

In the right triangle AED :

$$AE = \frac{DE}{\tan 48^\circ} = \frac{CF}{\tan 48^\circ} \approx 149.04 \text{ m}$$

So the area of the plot of land is

$$T = CF \cdot FE = CF \cdot (FA + AE) \approx 165.53 \cdot 219.30 \approx 36\,300 \text{ m}^2$$

Note: KoMaL multiplies values rounded to one decimal place and gets $36\,294 \text{ m}^2$; without the early rounding the area is about $36\,300 \text{ m}^2$.

Part II

5. Danny plays tennis. In a tournament, 63% of his first serves landed in. When his first serve was in, he won 90% of the points. When it was not in, he could serve a second time; 85% of these landed in. When his second serve was in, he won 60% of the points. When neither of his serves landed in, Danny lost the point.
- What is the probability that Danny lost a point when he was serving?
 - What is the probability that Danny's first serve was in, given that he lost the point?

(16 points)

Solution: a) Let A be the event that Danny wins a point, B_1 the event that his first serve is in, and B_2 the event that his second serve is in. The probability that Danny won the point after his first serve is

$$P(A | B_1) \cdot P(B_1) = 0.9 \cdot 0.63 = 0.567$$

He served a second time only in the 37% of the points where his first serve was not in. The probability that he won the point after his second serve is

$$P(A | B_2) \cdot P(B_2) \cdot P(\overline{B_1}) = 0.6 \cdot 0.85 \cdot 0.37 = 0.1887$$

The probability that Danny won the point is the sum of these two values, $P(A) = 0.7557$, so the probability that he lost the point is

$$P(\overline{A}) = 1 - P(A) = 0.2443$$

- b) We need the probability $P(B_1 | \overline{A})$. By the definition of conditional probability,

$$P(B_1 | \overline{A}) = \frac{P(B_1 \overline{A})}{P(\overline{A})} = \frac{P(\overline{A} | B_1) \cdot P(B_1)}{P(\overline{A})} = \frac{0.1 \cdot 0.63}{0.2443} \approx 0.258$$

6. a) Find the value of $\frac{x}{y}$ to three decimal places, if we know that $2^x = 5^y$, where x and y are nonzero real numbers.
b) Solve the following inequality on the largest possible subset of the real numbers.

$$\frac{5}{2} \sqrt{\lg x} + 2 \lg \sqrt{\frac{1}{x}} - 1 < 0$$

(16 points)

Solution: a) Taking the base 10 logarithm of both sides and using the well-known properties of logarithms: $\lg 2^x = \lg 5^y$, so $x \lg 2 = y \lg 5$. Dividing,

$$\frac{x}{y} = \frac{\lg 5}{\lg 2} \approx 2.322$$

- b) First we find the domain of the inequality. Because of the logarithms, $x > 0$ (and then $\sqrt{\frac{1}{x}} > 0$ too). Because of the square root, $\lg x \geq 0$, so $x \geq 1$. So the domain of the inequality is $x \geq 1$.

Using the properties of powers and logarithms, we transform the middle term of the left side:

$$2 \lg \sqrt{\frac{1}{x}} = 2 \lg x^{-\frac{1}{2}} = 2 \cdot \left(-\frac{1}{2}\right) \lg x = -\lg x$$

With the new unknown $a = \sqrt{\lg x}$ (which is nonnegative), the inequality becomes $-a^2 + 2.5a - 1 < 0$. The solutions of the corresponding equation are $a = 2$ and $a = 0.5$, and since the leading coefficient is negative, the solution of the inequality is $a < 0.5$ or $a > 2$.

Since a is nonnegative, we must solve the inequalities $0 \leq \sqrt{\lg x} < 0.5$ and $\sqrt{\lg x} > 2$. Squaring them, we get $0 \leq \lg x < 0.25$ and $\lg x > 4$. So

$$1 \leq x < \sqrt[4]{10} \quad (\approx 1.778) \quad \text{or} \quad x > 10\,000$$

Our transformations were equivalent.

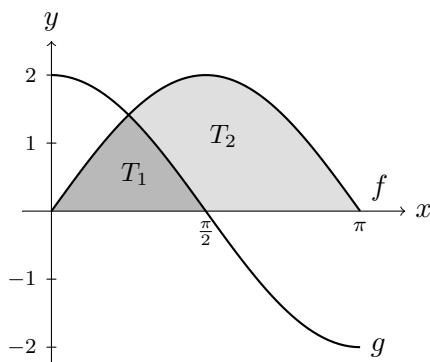
7. The region enclosed by the graph of the function $f(x) = 2 \sin x$ and the x -axis on the interval $[0, \pi]$ is divided into two regions by the graph of the function $g(x) = 2 \cos x$. Find the ratio of the areas of the two regions.

(16 points)

Solution: The area of the whole region is the integral of f from 0 to π :

$$T = \int_0^{\pi} 2 \sin x \, dx = [-2 \cos x]_0^{\pi} = 2 - (-2) = 4$$

The zero of g in the given interval is where $\cos x = 0$, that is, at $x = \frac{\pi}{2}$. The intersection of the two graphs is given by the solution of the equation $2 \sin x = 2 \cos x$; dividing by $\cos x \neq 0$, we get $\tan x = 1$, that is, $x = \frac{\pi}{4}$ in the given interval.



So

$$\begin{aligned} T_1 &= \int_0^{\frac{\pi}{4}} 2 \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2 \cos x \, dx = [-2 \cos x]_0^{\frac{\pi}{4}} + [2 \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= (-\sqrt{2} + 2) + (2 - \sqrt{2}) = 4 - 2\sqrt{2} \end{aligned}$$

and $T_2 = T - T_1 = 2\sqrt{2}$. The ratio of the areas of the two regions is

$$\frac{T_1}{T_2} = \frac{4 - 2\sqrt{2}}{2\sqrt{2}} = \frac{4\sqrt{2} - 4}{4} = \sqrt{2} - 1 \approx 0.414$$

8. The diagonals of a cyclic quadrilateral are perpendicular to each other. The intersection point of the diagonals is $E(2, 1)$, and the equations of the sides AB and BC of the quadrilateral are $x + 3y = 10$ and $x - y + 2 = 0$. Find the coordinates of the vertices of the quadrilateral and the equation of its circumcircle.

(16 points)

Solution: Solving the system formed by the equations of the two given sides, we get the point B . From the equation of BC , $y = x + 2$; substituting into the equation of AB , $x + 3(x + 2) = 10$, so $x = 1$ and $y = 3$: $B(1, 3)$.

A direction vector of the diagonal BD is $\overrightarrow{EB} = (-1, 2)$; with this and the point E , the equation of the diagonal BD is $2x + y = 5$. The diagonal AC is perpendicular to BD , so $(-1, 2)$ is a normal vector of it, and with the point E its equation is $-x + 2y = 0$.

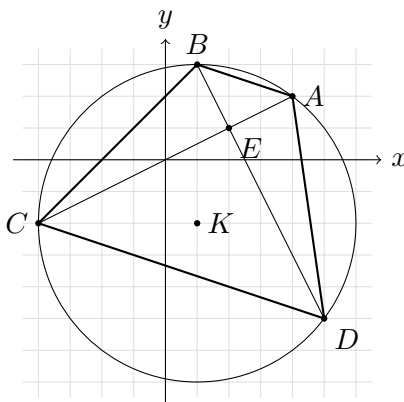
We get A by solving the system of the equations of the diagonal AC and the side AB . From the diagonal, $x = 2y$; substituting into AB , $5y = 10$, so $y = 2$, $x = 4$: $A(4, 2)$.

We get C by solving the system of the equations of the diagonal AC and the side BC : $2y - y + 2 = 0$, so $y = -2$, $x = -4$: $C(-4, -2)$.

The circumcircle of the quadrilateral is the circumcircle of the triangle ABC . Its center K is the intersection of the perpendicular bisectors of AC and BC . The midpoint of AC is the origin, and a normal vector of its perpendicular bisector is $(2, 1)$, so its equation is $2x + y = 0$. The midpoint of BC is $(-1.5, 0.5)$, and a normal vector of its perpendicular bisector is $\overrightarrow{CB} = (5, 5)$, so its equation is $x + y + 1 = 0$. Solving these, $K(1, -2)$. The radius of the circle is, for example, $KB = 5$, so the equation of the circle is

$$(x - 1)^2 + (y + 2)^2 = 25$$

Finally, we get D from the system of the equations of the circle and the diagonal BD . From BD , $y = 5 - 2x$; substituting into the equation of the circle, $(x - 1)^2 + (7 - 2x)^2 = 25$, that is, $5x^2 - 30x + 25 = 0$. Its solutions are $x = 1$ (this gives the point B) and $x = 5$, with $y = -5$: $D(5, -5)$.



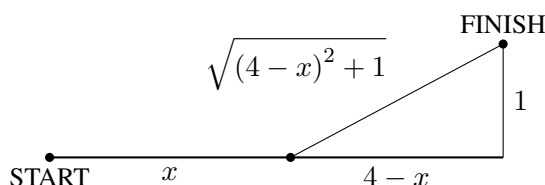
So the vertices are $A(4, 2)$, $B(1, 3)$, $C(-4, -2)$, $D(5, -5)$, and the circumcircle is $(x - 1)^2 + (y + 2)^2 = 25$.

9. The start of a duathlon race is on a straight sea shore, and the finish is in the water. One way to complete the distance is to first run 4 km along the shore, then turn 90° and swim 1 km out to sea. However, it is allowed to leave the running track at any time and start swimming toward the finish. Our speed is 6 km/h on the sandy shore and 2 km/h in the water. Measured from the start line, at what distance should we head into the water,
- if we want to cover equal distances running and swimming;
 - if we want to reach the finish as soon as possible? How many km do we have to swim then, and in how much time can we complete the race this way?

Give the answers to the nearest meter.

(16 points)

Solution: Let x (in km) be the distance from the start line where we head into the water. Then we run x km, and in the water we cover the hypotenuse of the right triangle in the figure, that is, $\sqrt{(4 - x)^2 + 1}$ km.



- We must solve the equation $x = \sqrt{(4 - x)^2 + 1}$. Both sides are positive, so we can square: $x^2 = (4 - x)^2 + 1$, so $x = \frac{17}{8}$. So we must head into the water 2.125 km, that is, 2125 m from the start line. Then the distance swum is indeed the same, since

$$\sqrt{\left(4 - \frac{17}{8}\right)^2 + 1} = \sqrt{\left(\frac{15}{8}\right)^2 + \left(\frac{8}{8}\right)^2} = \frac{17}{8}$$

b) The time needed to complete the distance is given by the function

$$t(x) = \frac{x}{6} + \frac{\sqrt{17 - 8x + x^2}}{2} \quad (0 < x < 4)$$

We look for its minimum, which can only be where its derivative is zero. By the chain rule,

$$t'(x) = \frac{1}{6} + \frac{2x - 8}{4\sqrt{17 - 8x + x^2}} = 0$$

Simplifying and rearranging: $12 - 3x = \sqrt{17 - 8x + x^2}$. Because of the condition on x , the left side is positive, so we can square; rearranging, we get $8x^2 - 64x + 127 = 0$. Its solutions are $x = 4 \pm \frac{\sqrt{2}}{4}$, of which only the one less than 4 is relevant: $x = 4 - \frac{\sqrt{2}}{4} \approx 3.646$. The function t is continuous, and one can check that the sign of its derivative changes from negative to positive at this x , so it is indeed a minimum of t .

So it is worth starting to swim after 3646 m measured from the start line. Then the distance swum is

$$\sqrt{\left(\frac{\sqrt{2}}{4}\right)^2 + 1} = \sqrt{\frac{18}{16}} = \frac{3\sqrt{2}}{4} \approx 1.061 \text{ km, that is, 1061 m}$$

and the time needed to complete the race is

$$t\left(4 - \frac{\sqrt{2}}{4}\right) = \frac{4}{6} - \frac{\sqrt{2}}{24} + \frac{3\sqrt{2}}{8} = \frac{4 + 2\sqrt{2}}{6} \approx 1.138 \text{ hours} \approx 68.3 \text{ minutes}$$

(If we ran the whole 4 km, we would complete the race in 70 minutes.)

Remark: Using Snell's law of refraction, the critical angle of total internal reflection also gives the solution of the problem.