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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The function $f(x) = \frac{x-671}{x-4} - \frac{x-503}{x-3}$ is defined on the set $\mathbb{R} \setminus \{3, 4\}$.

a) Find the zeros of the function f .

b) Is -417 in the range of f ? If yes, give the corresponding elements of the domain.

(11 points)

Solution: a) We look for the values of x for which $f(x) = 0$. Writing the fractions on the left side over a common denominator:

$$\frac{(x-671)(x-3) - (x-503)(x-4)}{(x-4)(x-3)} = 0$$

A fraction is zero when its numerator is zero:

$$(x^2 - 674x + 2013) - (x^2 - 507x + 2012) = 0, \quad -167x + 1 = 0, \quad x = \frac{1}{167}$$

So the only zero of f is $\frac{1}{167}$.

b) Now we look for the values of x for which $f(x) = -417$. As above,

$$\begin{aligned} \frac{-167x + 1}{(x-4)(x-3)} &= -417 \\ -167x + 1 &= -417(x-4)(x-3) \\ 417x^2 - 3086x + 5005 &= 0 \end{aligned}$$

so

$$x_{1,2} = \frac{3086 \pm \sqrt{3086^2 - 4 \cdot 417 \cdot 5005}}{2 \cdot 417} = \frac{3086 \pm 1084}{834}, \quad x_1 = 5, \quad x_2 = \frac{1001}{417}$$

So -417 is in the range of f , and there are two corresponding elements of the domain: $f(5) = -417$ and $f\left(\frac{1001}{417}\right) = -417$.

2. The annual subscription fee of a magazine was 1750 Ft. For the next year it was raised to 1800 Ft.

a) By what percent was it raised?

To tell the whole truth: when the magazine was cheaper, it appeared five times a year, with 32, 32, 32, 32, and 28 pages. When it became more expensive, only four issues came out per year, with 36, 48, 40, and 44 pages.

b) What change did this really mean for the subscribers, per page?

c) By how much did the number of pages of an issue change on average, comparing the two years?

d) Give the standard deviation of the numbers of pages of the issues for both years.

(13 points)

Solution: a) Since $\frac{1800}{1750} \approx 1.029$, the subscription fee was raised by about 3%.

b) Before the raise the subscribers got $32 + 32 + 32 + 32 + 28 = 156$ pages, and after it $36 + 48 + 40 + 44 = 168$ pages. Before the raise one page cost $\frac{1750}{156} \approx 11.22$ Ft on average; after the raise one page cost $\frac{1800}{168} \approx 10.71$ Ft on average. So per page, the price decreased. Since $\frac{10.71}{11.22} \approx 0.955$, this is a price decrease of about 4.5%.

c) Before the raise an issue had $\frac{156}{5} = 31.2$ pages on average, and after it $\frac{168}{4} = 42$ pages. So an issue became 10.8 pages (about 11 pages) longer on average.

d) We use the formula for the standard deviation:

$$D = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}}$$

where $\bar{x}$ is the mean. In the first year the mean was 31.2, and the standard deviation

$$D_1 = \sqrt{\frac{4 \cdot (32 - 31.2)^2 + (28 - 31.2)^2}{5}} = 1.6$$

In the second year the mean was 42, and the standard deviation

$$D_2 = \sqrt{\frac{(36 - 42)^2 + (48 - 42)^2 + (40 - 42)^2 + (44 - 42)^2}{4}} = \sqrt{20} \approx 4.47$$

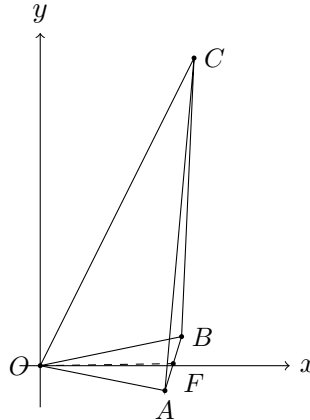
3. Three points are given in the coordinate system:

$$A(980, -1), \quad B(981, 1), \quad C(2^{1986}, 2 \cdot 2^{1986})$$

Reflecting the point C in the midpoint of AB , we get the point D . Find the area of the quadrilateral determined by the points A , B , C , and D .

(13 points)

Solution: The midpoint of AB is $F(980.5, 0)$. Because of the point reflection, $ACBD$ is a parallelogram, and its area is twice the area of the triangle ABC . Since the second coordinate of C is twice the first, the slope of the line OC (O is the origin) is 2. From the given coordinates, the slope of the line AB is also 2.



So AB and OC are parallel lines, and any point of the line OC forms with A and B a triangle whose area equals the area of the triangle ABC (since these triangles have the same altitude to the side AB). Of these triangles we compute the area of ABO (the figure is not to scale). OF is a median of this triangle, so its area is twice the area of the triangle FBO . Since $OF = 980.5$ and the altitude to this side is 1,

$$t_{FBO} = \frac{980.5 \cdot 1}{2}, \quad t_{ABC} = t_{ABO} = 2t_{FBO} = 980.5$$

The area of the quadrilateral is $T = 2t_{ABC} = 1961$.

Remark: Of course the area can also be found by the usual calculation from the given coordinates, for example from the distance of the point C from the line AB .

4. The lateral surface of a right circular cone, unrolled, is a quarter circle of radius 10 cm.

- What is the surface area and the volume of the cone?
- The cone is cut into two parts by a plane parallel to the base, at half the height. What is the ratio of the volumes of the two parts?

(14 points)

Solution: a) The length of the quarter circle arc of radius 10 cm is the circumference of the base circle of the cone:

$$i = \frac{2 \cdot 10\pi}{4} = 5\pi$$

Let the radius of the base circle be r ; then $5\pi = 2r\pi$, so $r = 2.5$ cm. The radius of the quarter circle is the slant height a of the cone. We know that $a^2 = r^2 + m^2$, where m is the height of the cone:

$$m = \sqrt{a^2 - r^2} = \sqrt{10^2 - 2.5^2} = \sqrt{93.75}$$

So the surface area and the volume are

$$\begin{aligned} A &= r^2\pi + r\pi a = 2.5^2\pi + 2.5 \cdot 10\pi \approx 98.2 \text{ cm}^2 \\ V &= \frac{r^2\pi m}{3} = \frac{2.5^2\pi\sqrt{93.75}}{3} \approx 63.4 \text{ cm}^3 \end{aligned}$$

b) The small cone cut off is similar to the original one, with ratio of similarity $\frac{1}{2}$, so the ratio of their volumes is $(\frac{1}{2})^3 = \frac{1}{8}$. So the small cone on top is $\frac{1}{8}$ of the whole, and the frustum below is $\frac{7}{8}$ of the whole. The ratio of the two parts is 1 : 7.

Part II

5. a) Prove that

$$a = \sqrt[3]{15 + 4\sqrt{14}} + \sqrt[3]{15 - 4\sqrt{14}}$$

is a root of the equation $x^3 - 3x - 30 = 0$.

b) Prove that the only root of the equation $x^3 - 3x - 18 = 0$ is

$$b = \sqrt[3]{9 + 4\sqrt{5}} + \sqrt[3]{9 - 4\sqrt{5}}$$

(16 points)

Solution: a) We take the cube of a , using $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$:

$$a^3 = 15 + 4\sqrt{14} + 15 - 4\sqrt{14} + 3\sqrt[3]{15 + 4\sqrt{14}}\sqrt[3]{15 - 4\sqrt{14}}\left(\sqrt[3]{15 + 4\sqrt{14}} + \sqrt[3]{15 - 4\sqrt{14}}\right)$$

The expression in parentheses equals a , and

$$\sqrt[3]{15 + 4\sqrt{14}}\sqrt[3]{15 - 4\sqrt{14}} = \sqrt[3]{(15 + 4\sqrt{14})(15 - 4\sqrt{14})} = \sqrt[3]{225 - 16 \cdot 14} = 1$$

so $a^3 = 30 + 3a$, that is, $a^3 - 3a - 30 = 0$. This shows exactly that a is a root of the equation $x^3 - 3x - 30 = 0$.

b) A calculator suggests that $b = 3$ (and 3 is indeed a root of $x^3 - 3x - 18 = 0$). In the same way as above,

$$b^3 = 9 + 4\sqrt{5} + 9 - 4\sqrt{5} + 3\sqrt[3]{(9 + 4\sqrt{5})(9 - 4\sqrt{5})} \cdot b = 18 + 3b$$

since $\sqrt[3]{81 - 16 \cdot 5} = 1$. So $b^3 - 3b - 18 = 0$, that is, b is a root of the equation $x^3 - 3x - 18 = 0$.

The left side of the cubic equation can be factored with the factor $x - 3$: $(x - 3)(x^2 + 3x + 6) = 0$. The second factor has no real zero, since its discriminant is negative. So the equation has only one real root, which is therefore b (and it equals 3).

Remark: The problem can also be solved using the perfect cubes $\left(\frac{3+\sqrt{5}}{2}\right)^3 = 9 + 4\sqrt{5}$ and $\left(\frac{3-\sqrt{5}}{2}\right)^3 = 9 - 4\sqrt{5}$.

6. In her history lesson Ibolya demonstrated indirect presidential elections to her students using the data of an imaginary country.

Bergengocia consists of eight states, so its official name is the United States of Bergengocia (USB for short). In Bergengocia the voters can choose between two parties, and then a given number of electors of the parties elect the president from one candidate of each of the two parties. In each state the number of electors is determined from the data of the last census. The current data are shown in the following table:

State	Voters	Electors
Aliforn	283 821	10
Lorida	234 171	9
Rizon	179 351	7
Olorad	38 021	3
Labama	28 221	3
Regon	26 991	3
Ebrasz	5 261	2
Laszka	4 171	2

The inhabitants of Bergengocia are very conscientious, so everybody exercises the right to vote. Everybody votes for the party whose electors they expect to elect the right president. The party that gets the most votes wins. In each state the winning party gives all the electors, and the electors can then freely choose between the two candidates (they are not bound by party discipline). In each state the presidential candidate supported by the majority of the electors gets the support of all the electors of the state. The candidate supported by the most electors wins the election.

- On average, how many voters live in each state of USB?
- In which state does it take the most, and in which the fewest votes to elect one elector?
- One of the candidates stated that he will surely be the winner, because he is certain of the support of three states. Which three states could he have had in mind?
- Could someone win this election if fewer than 20% of the voters wanted him to be the president?

(16 points)

Solution: a) We add the numbers of voters and divide by 8:

$$\frac{283\,821 + 234\,171 + 179\,351 + 38\,021 + 28\,221 + 26\,991 + 5\,261 + 4\,171}{8} = \frac{800\,008}{8} = 100\,001$$

So on average 100 001 voters live in each state of USB.

- We examine the states one by one. In each state, the integer part of half the number of voters plus 1 is needed for a party to win, and then this party gives all the electors. Dividing this number by the number of electors gives:

Aliforn	14 191.1
Lorida	13 009.6
Rizon	12 810.9
Olorad	6337
Labama	4703.7
Regon	4498.7
Ebrasz	1315.5
Laszka	1043

So it takes the most votes to elect one elector in Aliforn, and the fewest in Laszka.

- The number of electors is $10 + 9 + 7 + 3 + 3 + 3 + 2 + 2 = 39$, so a candidate needs the support of at least 20 electors to win.

If Aliforn and Lorida support him (19 electors), then the third state can be any of the others (each has more than 1 elector). If Aliforn and Rizon support him (17 electors) but Lorida does not, then the third state must have at least 3 electors, so only Olorad, Labama, or Regon can be the third. It is easy to see that there are no other possibilities (for example, $9 + 7 + 3 = 19$ is not enough). So the candidate could have had the following triples in mind:

Aliforn, Lorida, and any one of Rizon, Olorad, Labama, Regon, Ebrasz, Laszka;
Aliforn, Rizon, and any one of Olorad, Labama, Regon.

These are 9 triples in all.

d) If the candidate gets the support of Rizon, Olorad, Labama, Regon, Ebrasz, and Laszka, then he has 20 electors. In these states the parties need a majority of the votes, that is, everywhere at least one more voter than the integer part of half of the voters must vote for his party. Adding up these minimum values, this is

$$89\,676 + 19\,011 + 14\,111 + 13\,496 + 2631 + 2086 = 141\,011$$

votes. Even if nobody in the other states wanted him to win, he could win the election with these 20 electors. So of the 800 008 voters, 141 011 can be enough to win. Since $\frac{141\,011}{800\,008} \approx 0.176$, this means a support of only 17.6%.

So someone could win this election even if fewer than 20% of the voters wanted him to be the president.

Remark: Surprisingly, if the inhabitants of USB were not so conscientious, it could even happen that in these states only half of the voters exercised their right to vote. Then half of the active voters would be enough to get the 20 electors, which is about 70 500 people. That would be a support of hardly 9%, and the candidate could still win the election.

7. The price of a coat was originally 20 000 Ft, but the store started selling it at a discount. Shortly afterwards, in a new sale, its price became 14 960 Ft. In both cases the discount, expressed in percent, was an integer between 10 and 20. Laszlo bought a coat after the first discount. How much could he have paid then?

(16 points)

Solution: Let the first discount be $p\%$ and the second $q\%$. We know that $10 < p < 20$ and $10 < q < 20$, and both are integers. By the text,

$$20\,000 \left(1 - \frac{p}{100}\right) \left(1 - \frac{q}{100}\right) = 14\,960, \quad \text{that is,} \quad (100 - p)(100 - q) = 7480$$

Because of the possible values of p and q , we must write 7480 as a product of two integers between 80 and 90. The prime factorization helps: $7480 = 2^3 \cdot 5 \cdot 11 \cdot 17$. Since $11 \cdot 17$ is already a three-digit number, these prime factors cannot be in the same factor. If only factors 2 were written next to 17, it would not be suitable: $2 \cdot 17 < 80$, $2^2 \cdot 17 < 80$, $2^3 \cdot 17 > 90$. If both 5 and 2 were next to 17, the factor would again be too large, since $2 \cdot 5 \cdot 17 > 90$.

So the only possibility is that one factor is $100 - p = 5 \cdot 17 = 85$. Then the other factor is also suitable: $100 - q = 2^3 \cdot 11 = 88$. (Of course the values of the two factors can be swapped.) So one discount was 15% and the other 12%.

We do not know the order of the discounts, so the first discounted price could have been 85% of 20 000 Ft, that is, 17 000 Ft, or 88% of 20 000 Ft, that is, 17 600 Ft. So Laszlo bought the coat for either 17 000 Ft or 17 600 Ft. (One can check that 88% of 17 000 Ft and 85% of 17 600 Ft are both 14 960 Ft.)

8. a) The side lengths of a right triangle are consecutive terms of a geometric sequence, and its shortest side is 1 unit long. Compute the lengths of the other two sides of the triangle.
- b) The side lengths of a triangle are consecutive terms of a geometric sequence, and its shortest side is 1 unit long. We know that the triangle is not equilateral. Prove that the triangle has no 60° angle.

(16 points)

Solution: a) Let the common ratio of the geometric sequence be q . Then the legs of the right triangle are 1 and q , and its hypotenuse is q^2 . Of course $q > 1$, since the shortest side of the triangle is 1. By the Pythagorean theorem,

$$1 + q^2 = q^4, \quad q^4 - q^2 - 1 = 0$$

This is a quadratic equation in q^2 : $q^2 = \frac{1+\sqrt{5}}{2}$ or $q^2 = \frac{1-\sqrt{5}}{2}$. Clearly only $q^2 = \frac{1+\sqrt{5}}{2}$ is possible, so $q = \sqrt{\frac{1+\sqrt{5}}{2}}$. So the sides of the right triangle, in increasing order, are

$$1, \quad \sqrt{\frac{1+\sqrt{5}}{2}}, \quad \frac{1+\sqrt{5}}{2}$$

b) Let the common ratio be q ; we know that $q > 1$, since the shortest side is 1. Then the sides of the triangle are 1, q , q^2 . If the triangle is not equilateral and has a 60° angle, then this angle is the middle one in size, so it is opposite the middle side. We show that this is impossible. By the law of cosines for the triangle with sides 1, q , q^2 , with a 60° angle opposite q :

$$q^2 = 1 + q^4 - 2q^2 \cos 60^\circ = 1 + q^4 - q^2, \quad 0 = q^4 - 2q^2 + 1 = (q^2 - 1)^2$$

This gives only $q = -1$ and $q = 1$; the equation has no solution with $q > 1$. So the triangle has no 60° angle.

9. Consider the function f defined on the set of real numbers by $f(x) = x^3 + (p-4)x^2 + p^2x - 2$, where p is a real parameter.

- a) Compute the definite integral $\int_0^4 f(x) dx$, if $p = 4$.
- b) Find the value of p for which $x = 2$ is a zero of the function f .
- c) Find the integers p for which the derivative of the function f has no zero.

(16 points)

Solution: a) If $p = 4$, then $f(x) = x^3 + 16x - 2$, and the definite integral is

$$\int_0^4 (x^3 + 16x - 2) dx = \left[\frac{x^4}{4} + 8x^2 - 2x \right]_0^4 = 64 + 128 - 8 = 184$$

b) We look for the parameter p for which

$$f(2) = 2^3 + (p-4) \cdot 2^2 + p^2 \cdot 2 - 2 = 0, \quad \text{that is,} \quad 2p^2 + 4p - 10 = 0, \quad p^2 + 2p - 5 = 0$$

By the quadratic formula we found two suitable parameters: $p_1 = -1 + \sqrt{6}$ and $p_2 = -1 - \sqrt{6}$.

c) The derivative of f is $f'(x) = 3x^2 + 2(p-4)x + p^2$. This quadratic function has no zero exactly when its discriminant is negative:

$$D = 4(p-4)^2 - 12p^2 < 0, \quad -8p^2 - 32p + 64 < 0, \quad p^2 + 4p - 8 > 0$$

The zeros of the quadratic expression $p^2 + 4p - 8$ are $-2 - 2\sqrt{3} \approx -5.46$ and $-2 + 2\sqrt{3} \approx 1.46$, and it is positive outside the interval between them. So the integers in question are all integers $p \leq -6$ and all integers $p \geq 2$.

Note: KoMaL's solution looks for the parameters for which the derivative has two distinct zeros (positive discriminant), and gets $-5, -4, -3, -2, -1, 0, 1$; these are the answer to the opposite question. (For p in this list the derivative has two zeros; for the problem as stated, the answer is the complementary set of integers.)