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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. When Johnny crossed the finish line in a cross-country race, the reporter on the spot asked him which place he had finished in. Johnny answered: "If half of those who finished ahead of me had finished behind me, then five times as many would have finished behind me as ahead of me. But if a third of those who finished behind me had finished ahead of me, then 6 more would have finished ahead of me than behind me."

How many runners took part in the race, and which place did Johnny finish in?

(12 points)

Solution: Let x be the number of runners who finished ahead of Johnny, and y the number who finished behind him. By the first condition,

$$y + \frac{x}{2} = 5 \left(x - \frac{x}{2} \right), \quad \text{that is,} \quad y = 2x$$

By the second condition,

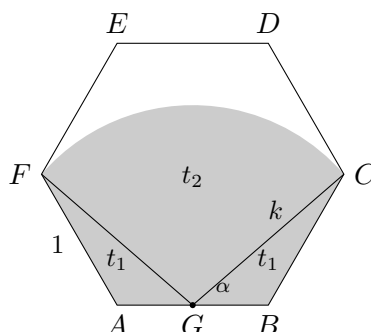
$$x + \frac{y}{3} = \left(y - \frac{y}{3} \right) + 6, \quad \text{that is,} \quad \frac{y}{3} = x - 6$$

From the two equations, $\frac{2x}{3} = x - 6$, so $x = 18$ and $y = 36$. So 18 runners finished ahead of Johnny and 36 behind him. The number of runners was $18 + 1 + 36 = 55$, and Johnny finished in 19th place.

2. A goat is tied to a stake driven into the midpoint of one side of a meadow in the shape of a regular hexagon. The rope is just long enough (when it is taut) for the goat to reach the far endpoints of the two neighboring sides. What percent of the meadow can the goat graze?

(12 points)

Solution: Let the side of the regular hexagon $ABCDEF$ be 1 unit, and let the goat be tied at the midpoint G of the side AB . Then the area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2} \approx 2.598$, and the length of the rope is $k = GF = GC$.



The region the goat can reach consists of two congruent triangles and a circular sector. The area of each of the triangles AGF and BGC is

$$t_1 = \frac{1 \cdot \frac{1}{2} \cdot \sin 120^\circ}{2} = \frac{\sqrt{3}}{8}$$

By the law of cosines in the triangle AGF ,

$$k^2 = 1 + \frac{1}{4} - 2 \cdot 1 \cdot \frac{1}{2} \cos 120^\circ = \frac{7}{4}, \quad \text{so} \quad k = \frac{\sqrt{7}}{2}$$

Let $\alpha = \angle AGF$. By the law of sines in the triangle AGF ,

$$\frac{\sin \alpha}{\sin 120^\circ} = \frac{1}{\frac{\sqrt{7}}{2}}, \quad \text{so} \quad \sin \alpha = \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{7}} = \sqrt{\frac{3}{7}}, \quad \text{and} \quad \alpha \approx 40.9^\circ$$

So the central angle of the sector is $\angle FGC = 180^\circ - 2\alpha \approx 98.2^\circ$, and the area of the sector is

$$t_2 = \frac{98.2^\circ}{360^\circ} \cdot k^2 \pi \approx 1.500$$

The area the goat can reach is $T = 2t_1 + t_2 \approx \frac{\sqrt{3}}{4} + 1.500 \approx 1.933$. So the goat can graze $\frac{1.933}{2.598} \approx 0.744$, that is, about 74.4% of the meadow.

3. The universal set is the set of the first n positive integers. Let A be the set of the numbers divisible by 3, B the set of the numbers divisible by 4, and C the set of the numbers divisible by 5.

- a) If $n = 100$, what is the probability that a randomly chosen number is not an element of any of the three sets?
b) If the set $A \cap B \cap C$ has 2 elements, how many elements can the set $(A \cap B) \setminus C$ have?

(13 points)

Solution: a) The set $A \cap B \cap C$ has only 1 element (namely 60). The elements of A are the positive integers not greater than 100 divisible by 3; they are consecutive terms of an arithmetic sequence with first term 3, last term 99, and common difference 3, so A has 33 elements. Similarly, B has 25 and C has 20 elements.

$A \cap B$ consists of the numbers divisible by 12: $\{12, 24, 36, 48, 60, 72, 84, 96\}$, 8 elements. $B \cap C$ consists of the numbers divisible by 20: $\{20, 40, 60, 80, 100\}$, 5 elements. $A \cap C$ consists of the numbers divisible by 15: $\{15, 30, 45, 60, 75, 90\}$, 6 elements.

So the number of elements in at least one of the sets is $33 + 25 + 20 - (8 + 5 + 6) + 1 = 60$, and the number of elements in none of the sets is 40. The probability in question is $\frac{40}{100} = 0.4$.

b) If $A \cap B \cap C$ has 2 elements, these can only be 60 and 120, so $120 \leq n < 180$. The set $(A \cap B) \setminus C$ consists of the positive multiples of 12 not greater than n , except 60 and 120. These are the numbers $12k$ with $12k \leq n < 180$, $k \neq 5$, $k \neq 10$. The possible values of k are 1, 2, 3, 4, 6, 7, 8, 9 (these always), and 11, 12, 13, 14 (depending on the value of n). So the set $(A \cap B) \setminus C$ has 8, 9, 10, 11, or 12 elements.

4. According to the statistics of the Hungarian State Railways (MÁV), 8% of the passengers ride without a valid ticket.
- a) There are 24 passengers in a railway car. What is the probability that the conductor finds a passenger without a ticket in the car?
- b) For how many passengers is the probability that the conductor finds a passenger without a ticket in the car at least 90%?

(14 points)

Solution: a) The probability that a passenger has no ticket is 0.08, so the probability that a passenger has a valid ticket is 0.92. The probability that the conductor finds no passenger without a ticket is $0.92^{24} \approx 0.1352$. So the probability that there is a passenger without a ticket among the 24 passengers is $1 - 0.1352 = 0.8648$, about 86.5%.

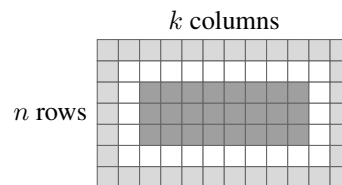
b) The probability that the conductor finds a passenger without a ticket is at least 90% exactly when the probability that he does not find one is at most 10%. For n passengers this means $0.92^n \leq 0.1$. Taking the base 10 logarithm of both sides, $n \cdot \lg 0.92 \leq \lg 0.1$, and since $\lg 0.92 < 0$,

$$n \geq \frac{\lg 0.1}{\lg 0.92} \approx 27.62$$

So if there are at least 28 passengers, the probability that the conductor finds a passenger without a ticket in the car is at least 90%.

Part II

5. A grid of unit squares has n rows and k columns.



The number of squares touching the edge of the grid (light gray in the figure), the number of squares touching those squares (white in the figure), and the number of squares inside the grid (dark gray in the figure) are consecutive terms of an arithmetic sequence. Find the values of n and k .

(16 points)

Solution: The conditions imply $n > 4$ and $k > 4$. The number of squares on the edge of the grid is $a_1 = 2k + 2n - 4$. The number of squares touching these is $a_2 = 2(k - 2) + 2(n - 2) - 4 = 2k + 2n - 12$. The number of all other squares is $a_3 = (k - 4)(n - 4)$.

The common difference of the arithmetic sequence is $a_2 - a_1 = -8$. So

$$a_3 - a_2 = (k - 4)(n - 4) - 2k - 2n + 12 = -8$$

that is, $kn - 6n - 6k + 36 = 0$, so $(k - 6)(n - 6) = 0$. This holds exactly when $k = 6$ or $n = 6$.

So the conditions of the problem are satisfied if one side of the grid is 6, and the other is any positive integer greater than 4.

6. For which positive integers n is the value of the fraction $\frac{3\sqrt{n}+\sqrt{3}}{4\sqrt{3}-\sqrt{n}}$ a positive integer?

(16 points)

Solution: Let k be the positive integer with

$$\begin{aligned}\frac{3\sqrt{n} + \sqrt{3}}{4\sqrt{3} - \sqrt{n}} &= k \\ 3\sqrt{n} + \sqrt{3} &= 4k\sqrt{3} - k\sqrt{n} \\ \sqrt{n}(3 + k) &= \sqrt{3}(4k - 1) \\ \frac{4k - 1}{3 + k} &= \sqrt{\frac{n}{3}}\end{aligned}$$

The left side is a rational number. The right side is rational exactly when the number under the square root is the square of a rational number, that is, $n = 3r^2$, where r is a positive integer. Then $\frac{4k-1}{3+k} = r$. Rewrite the left side:

$$\frac{4k - 1}{3 + k} = \frac{4k + 12 - 13}{3 + k} = 4 - \frac{13}{3 + k}$$

This is an integer exactly when $3 + k$ is a divisor of 13, that is, $3 + k = 1, -1, 13$, or -13 . Of these, only $3 + k = 13$ gives a positive integer value for k , namely $k = 10$. Then $\frac{39}{13} = 3 = \sqrt{\frac{n}{3}}$, so $n = 27$.

7. The Budapest–Zürich international express train consists of 12 cars: 4 first-class cars, 6 second-class cars, a dining car, and a baggage car.

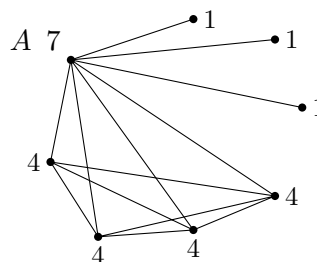
- a) In how many ways can the cars be ordered so that the first-class cars are next to each other, the second-class cars are next to each other, and the dining car is not the last car of the train?
- b) 8 scientists traveled in one of the cars, some of whom already knew each other. One of the scientists (let us call him A) knew everyone in the group. There were three scientists among them who knew nobody except A , while the other four knew each other. Those who did not know each other introduced themselves with a handshake. How many handshakes were there?

(16 points)

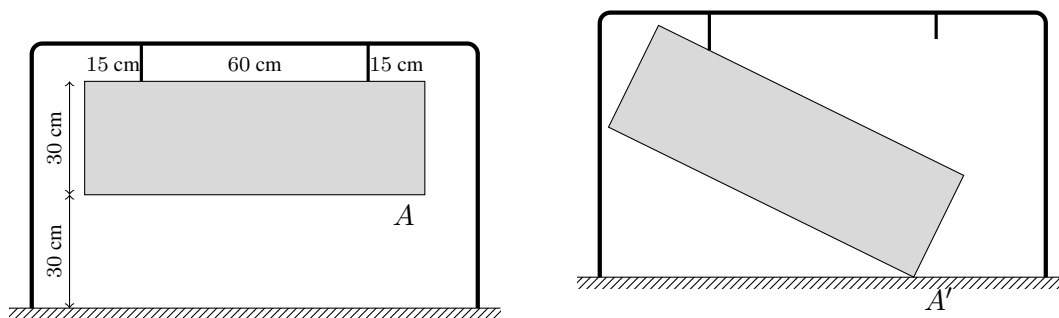
Solution: a) The 4 kinds of cars (the block of first-class cars, the block of second-class cars, the dining car, and the baggage car) can be put in order in $4!$ ways. Each of these permutations can be realized in $6! \cdot 4!$ ways, since the first-class cars can be ordered in $4!$ ways and the second-class cars in $6!$ ways. So there are $4! \cdot 4! \cdot 6!$ orders. From these we must subtract the cases where the dining car is the last car of the train; there are $3! \cdot 4! \cdot 6!$ of these. So the number of orders satisfying all conditions is

$$4! \cdot 4! \cdot 6! - 3! \cdot 4! \cdot 6! = 3! \cdot 4! \cdot 6! (4 - 1) = 311\,040$$

- b) Represent the acquaintances by a graph, and write the degree of each vertex next to it. The vertex of the scientist A has degree 7. Three people know nobody except A , so these three vertices have degree 1. The other four people all know each other (and A), so their vertices have degree 4.



8. In front of the entrance of a restaurant, vertical rods were welded to a metal frame standing on level ground, and the daily menu was posted on a board screwed to the rods. After a while one of the bearings broke (see the figure), and so the board tipped down to the ground.



- (16 points)

$$(x - 1.5)^2 + (y - 6)^2 = 65.25$$
$$(x - 1.5)^2 + 36 = 65.25, \quad (x - 1.5)^2 = 29.25, \quad x = 1.5 \pm \sqrt{29.25}$$
$$\tan \alpha = \frac{AB}{BK} = \frac{30}{75} = 0.4, \quad \text{so} \quad \alpha \approx 21.8^\circ$$

Let B' be the projection of A' onto the line KB . We get β from the right triangle $B'KA'$:

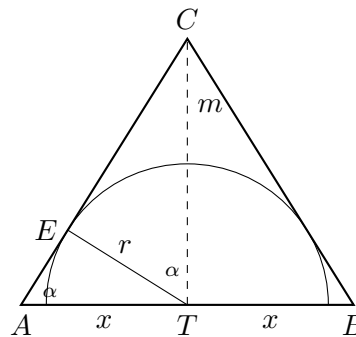
$$\tan \beta = \frac{A'B'}{B'K} \approx \frac{60}{69.1 - 15} = \frac{60}{54.1} \approx 1.109, \quad \text{so} \quad \beta \approx 48.0^\circ$$

So the board turned through about $48.0^\circ - 21.8^\circ = 26.2^\circ$ as it tipped.

9. A semicircle is drawn about the midpoint of the base of an isosceles triangle, tangent to the legs of the triangle. Find the angles of the triangle if the area of the semicircle is the largest possible percent of the area of the triangle.

(16 points)

Solution: We use the labels of the figure. Let α be the base angle of the triangle; then $\angle ETC = \alpha$, since $\angle EAT$ and $\angle ETC$ are acute angles with perpendicular sides.



In the right triangle EAT , $\sin \alpha = \frac{r}{x}$, so $x = \frac{r}{\sin \alpha}$. In the right triangle ETC , $\cos \alpha = \frac{r}{m}$, so $m = \frac{r}{\cos \alpha}$. The area of the triangle is

$$T_{ABC} = \frac{2xm}{2} = xm = \frac{r^2}{\sin \alpha \cos \alpha}$$

The area of the semicircle is $\frac{r^2\pi}{2}$. The ratio of the areas of the semicircle and the triangle, as a function of the acute angle α , is

$$H(\alpha) = \frac{\frac{r^2\pi}{2}}{\frac{r^2}{\sin \alpha \cos \alpha}} = \frac{\pi}{2} \sin \alpha \cos \alpha$$

It can have an extreme value where its derivative is 0:

$$H'(\alpha) = \frac{\pi}{2} (\cos^2 \alpha - \sin^2 \alpha) = 0$$

Since α is acute, $\sin \alpha = \cos \alpha$, so $\alpha = 45^\circ$. This is indeed a maximum, since $H'(\alpha) > 0$ for $\alpha < 45^\circ$ and $H'(\alpha) < 0$ for $\alpha > 45^\circ$.

So the angles of the triangle are 45° , 45° , and 90° . (Then the semicircle covers $\frac{\pi}{4} \approx 78.5\%$ of the triangle.)

Remark: Of course, using the addition formulas, we could have reached the same result without differentiation:

$$H(\alpha) = \frac{\pi}{2} \sin \alpha \cos \alpha = \frac{\pi}{4} \sin 2\alpha.$$