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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation over the real numbers.

$$x^2 + \frac{2x}{x+5} + \frac{1}{x^2 + 10x + 25} = 25$$

(12 points)

Solution: Because of the denominators, $x \neq -5$. The equation can be written in the form

$$\left(x + \frac{1}{x+5}\right)^2 = 25$$

There are two cases.

Case I: If $x + \frac{1}{x+5} = 5$, then after rearranging we get the quadratic equation $x^2 = 24$. Its roots are $x_1 = \sqrt{24} = 2\sqrt{6}$ and $x_2 = -2\sqrt{6}$, and both are solutions of the equation.

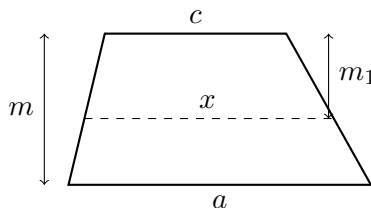
Case II: If $x + \frac{1}{x+5} = -5$, then after rearranging we get the quadratic equation $x^2 + 10x + 26 = 0$. Its discriminant is negative, so in this case there are no real roots.

So the equation has two solutions: $x_1 = 2\sqrt{6}$ and $x_2 = -2\sqrt{6}$.

2. The parallel sides of a trapezoid are a and c . How long is the segment parallel to the bases that bisects the area of the trapezoid?

(12 points)

Solution: Let x be the length of the segment, let m be the height of the trapezoid, and let m_1 be the height of the upper small trapezoid (with bases c and x).



The segment divides the trapezoid into two trapezoids, each with half of the area T of the whole:

$$\frac{a+c}{2} \cdot m = 2 \cdot \frac{x+c}{2} \cdot m_1 \quad \text{and} \quad \frac{a+c}{2} \cdot m = 2 \cdot \frac{x+a}{2} (m - m_1)$$

From the first equation $m_1 = \frac{a+c}{2(x+c)}m$. Substituting into the second equation and dividing by $m \neq 0$:

$$\begin{aligned} \frac{a+c}{2} &= (x+a) \left(1 - \frac{a+c}{2(x+c)}\right) \\ (a+c)(x+c) &= 2(x+a)(x+c) - (x+a)(a+c) \\ c^2 &= 2x^2 - a^2 \end{aligned}$$

(the last line after expanding and canceling). So the length of the segment is

$$x = \sqrt{\frac{a^2 + c^2}{2}}$$

3. Solve the equation $\log_x(5x^2) \cdot \log_5^2 x = 1$ over the real numbers.

(13 points)

Solution: The domain of the equation is $x > 0$, $x \neq 1$. Using the properties of logarithms, the equation can be written as

$$(\log_x 5 + \log_x x^2) \cdot \log_5^2 x = 1, \quad \text{that is,} \quad \left(\frac{1}{\log_5 x} + 2\right) \cdot \log_5^2 x = 1$$

Multiplying out and rearranging: $2\log_5^2 x + \log_5 x - 1 = 0$, so $\log_5 x = \frac{1}{2}$ or $\log_5 x = -1$. So the solutions of the equation are $x_1 = \sqrt{5}$ and $x_2 = \frac{1}{5}$.

4. A snack bar sells sandwiches and pizzas. The owner makes a 30% profit on a sandwich and a 50% profit on a pizza. One day the owner sold as many sandwiches as pizzas, and so made a 34% profit. The next day, however, customers bought twice as many pizzas as sandwiches.

- What was the owner's profit on the goods sold on the second day?
- What would the profit have been if the owner had sold twice as many sandwiches as pizzas?

(14 points)

Solution: a) Let a be the cost of a sandwich and b the cost of a pizza. If on the first day the owner sold x of each, then the profit was 34%, so

$$x \cdot 0.3a + x \cdot 0.5b = x(a+b) \cdot 0.34, \quad 0.16b = 0.04a, \quad a = 4b$$

The next day the owner sold y sandwiches and $2y$ pizzas. If the profit was $p\%$, then

$$y \cdot 0.3a + 2y \cdot 0.5b = (ya + 2yb) \cdot \frac{p}{100}, \quad \text{so} \quad \frac{p}{100} = \frac{0.3a + b}{a + 2b} = \frac{0.3 \cdot 4b + b}{4b + 2b} = \frac{2.2}{6}, \quad \text{and} \quad p = 36.\dot{6}$$

So the owner's profit was about 36.7%.

- If the owner had sold y sandwiches and half as many pizzas, then

$$y \cdot 0.3a + \frac{y}{2} \cdot 0.5b = \left(ya + \frac{yb}{2}\right) \cdot \frac{p}{100}, \quad \text{so} \quad \frac{p}{100} = \frac{0.3 \cdot 4b + 0.25b}{4b + 0.5b} = \frac{1.45}{4.5}, \quad \text{and} \quad p \approx 32.2$$

So the owner's profit would have been about 32.2%.

Part II

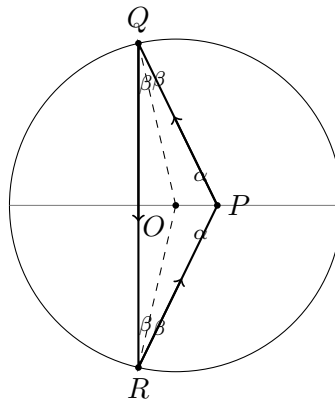
5. A billiard ball is at the point P of a circular billiard table of diameter 4 m, 0.5 meters from its center O . The ball must be shot so that after two bounces it passes through the point P again. What is the angle between the direction of the shot and the direction PO ?

(16 points)

Solution: The directions at angles 0° and 180° with PO satisfy the conditions (the ball moves along a diameter). Let us look for a direction at another angle α .

Starting from P , the ball bounces at the points Q and R and returns to P . By the law of reflection, $\angle PQO = \angle OQR$ and $\angle QRO = \angle ORP$. The triangle OQR is isosceles; let β be the angle of incidence and reflection. By symmetry, $\angle OPQ = \angle OPR = \alpha$, so in the triangle PQR ,

$$2\alpha + 4\beta = 180^\circ, \quad \text{so} \quad \beta = 45^\circ - \frac{\alpha}{2}$$



By the law of sines in the triangle OPQ , $\frac{\sin \alpha}{\sin \beta} = \frac{OQ}{OP} = \frac{2}{0.5} = 4$. So

$$\sin \alpha = 4 \sin \left(45^\circ - \frac{\alpha}{2} \right) = 4 \left(\frac{1}{\sqrt{2}} \cos \frac{\alpha}{2} - \frac{1}{\sqrt{2}} \sin \frac{\alpha}{2} \right)$$

Squaring (both sides are positive here):

$$\sin^2 \alpha = 8 \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)^2 = 8 \left(\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2} - 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right) = 8(1 - \sin \alpha)$$

So $\sin^2 \alpha + 8 \sin \alpha - 8 = 0$. Its roots are $\sin \alpha = -4 + \sqrt{24}$, which gives $\alpha \approx 64^\circ$, and $\sin \alpha = -4 - \sqrt{24} < -1$, which gives no solution.

So the angle in question is $\alpha \approx 64^\circ$ (besides the trivial directions along the diameter).

6. The general term of a sequence is given by the formula

$$a_n = \frac{2n^2 + 3n - 1}{n^2 + 4}$$

- a) Find the limit $\lim_{n \rightarrow \infty} a_n$.
 b) Find the threshold index N from which on the terms of the sequence differ from the limit of the sequence by less than 10^{-2} .

(16 points)

Solution: a)

$$A = \lim_{n \rightarrow \infty} \frac{2n^2 + 3n - 1}{n^2 + 4} = \lim_{n \rightarrow \infty} \frac{2 + \frac{3}{n} - \frac{1}{n^2}}{1 + \frac{4}{n^2}} = 2$$

- b) We must find the index from which on $|a_n - A| < \frac{1}{100}$, that is,

$$\left| \frac{2n^2 + 3n - 1}{n^2 + 4} - 2 \right| = \left| \frac{3n - 9}{n^2 + 4} \right| < \frac{1}{100}$$

For $n > 3$, $\frac{3n-9}{n^2+4} > 0$, so we must solve $\frac{3n-9}{n^2+4} < \frac{1}{100}$. Multiplying both sides by the common denominator and rearranging, $n^2 - 300n + 904 > 0$. The roots of the corresponding quadratic equation are $n_1 \approx 3.04$ and $n_2 \approx 296.96$. So the threshold index is $N = 297$.

7. The school cafeteria usually orders potatoes from two places in 10 kg bags. Twice as many are ordered from supplier A as from supplier B. Supplier A delivers the order within a week with probability 80%, and supplier B with probability 60%.
- a) What is the probability that for a randomly chosen 10 kg bag we must wait more than a week?
 b) If we know that the potatoes were delivered within a week, what is the probability that they were ordered from A?
 c) If in case of a late delivery the potatoes are sold at a 10% discount, and a 10 kg bag costs 800 Ft at both suppliers, then what is the expected price paid for a 10 kg bag?

(16 points)

Solution: a) Let x be the number of 10 kg bags ordered from supplier B. The expected number of orders arriving after more than a week is $0.2 \cdot 2x$ at A and $0.4x$ at B, a total of $0.8x$. The total number of orders is $3x$. So

$$P = \frac{0.2 \cdot 2x + 0.4x}{3x} = \frac{0.8}{3} \approx 0.27$$

- b) Let A be the event that the goods were ordered from A, and H the event that they arrived within a week. Then

$$P(A | H) = \frac{P(AH)}{P(H)} = \frac{0.8 \cdot 2x}{3x} : \frac{0.8 \cdot 2x + 0.6x}{3x} = \frac{1.6}{2.2} \approx 0.73$$

- c) The expected price is

$$M = \frac{0.8}{3} \cdot 800 \cdot 0.9 + \left(1 - \frac{0.8}{3}\right) \cdot 800 \approx 778.67$$

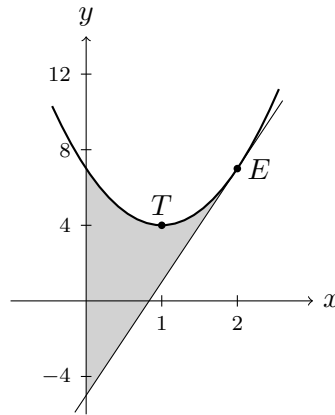
So the expected price of a bag is about 779 Ft.

Note: KoMaL uses the rounded probability 0.27 and gets 778.4 Ft.

8. The vertex of a parabola with axis parallel to the y -axis is $T(1, 4)$, and the slope of the tangent line at the point of the parabola with x -coordinate 2 is 6. Find the area of the plane region bounded by the y -axis, the parabola, and the tangent line at the point of the parabola with x -coordinate 2.

(16 points)

Solution: The equation of the parabola is $y = a(x - 1)^2 + 4 = ax^2 - 2ax + a + 4$, so $y' = 2ax - 2a$, and $6 = y'(2) = 2a$, so $a = 3$. The parabola is $y = 3x^2 - 6x + 7$, and the tangent line at $E(2, 7)$ is $y = 6x - 5$.



The region lies between the parabola (above) and the tangent line (below) for $0 \leq x \leq 2$, so its area is

$$t = \int_0^2 [(3x^2 - 6x + 7) - (6x - 5)] dx = \int_0^2 3(x - 2)^2 dx = \left[(x - 2)^3 \right]_0^2 = 8$$

So the area of the region is 8 square units.

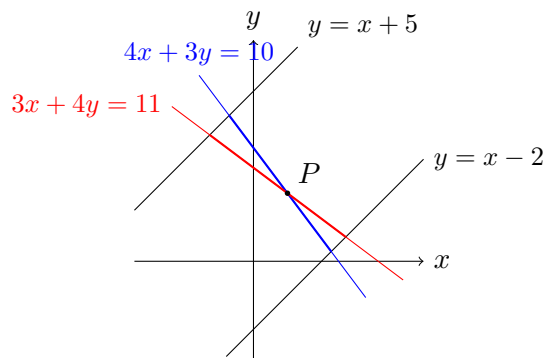
9. Find the equation of the line through the point $P(1, 2)$ whose segment between the lines $x - y + 5 = 0$ and $x - y = 2$ is 5 units long.

(16 points)

Solution: Let AB be a line satisfying the conditions, and translate it parallel to itself through the origin; let this be the line MN , with equation $y = mx$. (A vertical line cuts a segment of length 7 from the two lines, so it is not a solution.)

The intersection of the lines $y = mx$ and $y = x + 5$ is $M\left(\frac{5}{m-1}, \frac{5m}{m-1}\right)$, and the intersection of the lines $y = mx$ and $y = x - 2$ is $N\left(\frac{-2}{m-1}, \frac{-2m}{m-1}\right)$. By the condition $MN = 5$, so

$$MN^2 = 25 = \left(\frac{7}{m-1}\right)^2 + \left(\frac{7m}{m-1}\right)^2$$



Rearranging, we get the equation $12m^2 + 25m + 12 = 0$, whose solutions are $m = -\frac{3}{4}$ and $m = -\frac{4}{3}$. The lines with these slopes through the point $P(1, 2)$ are

$$4x + 3y = 10 \quad \text{and} \quad 3x + 4y = 11$$