

author: László Számadó

translated from: KoMaL, March 2012, pages 132-133

Pdf file: KoMaL, March 2012, pages 132-133

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Pharmacies sell a cold-relief powder, to be dissolved in hot water, in identical small sachets, packed six or ten to a cardboard box. A box of six costs 1130 Ft, and a box of ten costs 1640 Ft. Since the material cost of the boxes is minimal, assume that their price is the same regardless of their size, and that the price of the sachets of powder is also the same. What would a box of fourteen cost then?

(11 points)

Solution: Let the price of a sachet be x Ft and the price of a box y Ft. By the text,

$$\begin{aligned} 6x + y &= 1130 \\ 10x + y &= 1640 \end{aligned}$$

Subtracting the first equation from the second, $4x = 510$, so $x = 127.5$, and then $y = 365$. So a sachet costs 127.5 Ft and a box costs 365 Ft, under our assumptions. Then a box of fourteen would cost $14x + y = 14 \cdot 127.5 + 365 = 2150$ Ft.

2. Solve the following equation.

$$\lg(1-x) \cdot \sqrt{x^2 - 3x - 28} \cdot \cos\left(x + \frac{\pi}{6}\right) = 0$$

(13 points)

Solution: Because of the logarithm, $x < 1$, and because of the square root, $x^2 - 3x - 28 \geq 0$, that is, $x \leq -4$ or $x \geq 7$. So the domain of the equation is $x \leq -4$.

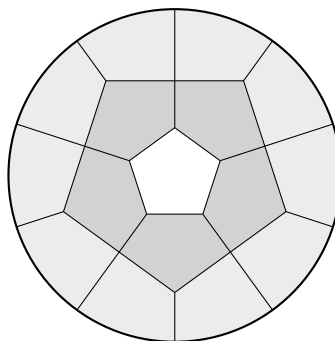
We look for the zeros of the factors one by one. The zero of the first factor is 0, but it is not in the domain. The zeros of the second factor are -4 and 7 , but only -4 is in the domain. The zeros of the third factor are

$$x + \frac{\pi}{6} = \frac{\pi}{2} + k\pi \quad (k \text{ an integer}), \quad \text{so} \quad x = \frac{\pi}{3} + k\pi$$

Because of the domain, however, $\frac{\pi}{3} + k\pi \leq -4$, that is, $k \leq \frac{-4 - \frac{\pi}{3}}{\pi} \approx -1.607$, so $k \leq -2$.

So the solutions are $x = -4$ and $x = \frac{\pi}{3} + k\pi$, where $k \leq -2$ is an integer.

3. Bettina bought the latest mosaic powder. The figure shows a sketch of the pattern of the cylindrical compact of radius 4 cm, seen from above.



- a) The pentagon in the middle of the pattern is regular, and together with the pentagons attached to it, it forms a larger regular pentagon. What would be the ratio of the sides of these two regular pentagons if the 16 regions of the compact had equal areas?
- b) Under the previous conditions, find the side length of the small regular pentagon in the middle to the nearest millimeter.

(13 points)

Solution: a) Since the small regions have equal areas, the ratio of the areas of the two regular pentagons is $\frac{1}{6}$ (the larger one consists of the small pentagon and the 5 pentagons attached to it). The ratio of the areas is the square of the ratio of similarity. Let a' be the side of the small pentagon and a the side of the large one. Then

$$\left(\frac{a'}{a}\right)^2 = \frac{1}{6}, \quad \text{so} \quad \frac{a'}{a} = \frac{1}{\sqrt{6}} \approx 0.4082$$

- b) The area of the circle is $4^2\pi = 16\pi$. The pattern consists of 16 regions of equal area, so the area of the regular pentagon in the middle is π . So we must find the side of a regular pentagon of area π .

A regular pentagon can be divided from its center into five isosceles triangles with apex angle 72° . Let the base of these triangles be $2x$; then the altitude to the base is $m = \frac{x}{\tan 36^\circ}$. So the area is

$$5 \cdot x \cdot \frac{x}{\tan 36^\circ} = \pi, \quad \text{so} \quad 2x = \sqrt{\frac{4}{5}\pi \tan 36^\circ} \approx 1.35$$

So the side of the small regular pentagon is about 1.35 cm, that is, 14 mm to the nearest millimeter.

4. Solve the equation

$$2 \sin^2 2x - 2 \cos^2 x = 1$$

(14 points)

Solution: Using the well-known identities, we transform the equation:

$$\begin{aligned} 2(2 \sin x \cos x)^2 - 2 \cos^2 x &= \sin^2 x + \cos^2 x \\ 8 \sin^2 x \cos^2 x - 3 \cos^2 x - \sin^2 x &= 0 \\ 8(1 - \cos^2 x) \cos^2 x - 3 \cos^2 x - (1 - \cos^2 x) &= 0 \\ 8 \cos^4 x - 6 \cos^2 x + 1 &= 0 \end{aligned}$$

So

$$\cos^2 x = \frac{6 \pm \sqrt{36 - 32}}{16} = \frac{6 \pm 2}{16}, \quad \text{that is,} \quad \cos^2 x = \frac{1}{2} \quad \text{or} \quad \cos^2 x = \frac{1}{4}$$

So $\cos x = \pm \frac{\sqrt{2}}{2}$ or $\cos x = \pm \frac{1}{2}$. Combining the solutions:

$$x = \frac{\pi}{4} + k \cdot \frac{\pi}{2}, \quad x = \frac{\pi}{3} + k\pi, \quad x = \frac{2\pi}{3} + k\pi, \quad \text{where } k \in \mathbb{Z}$$

Part II

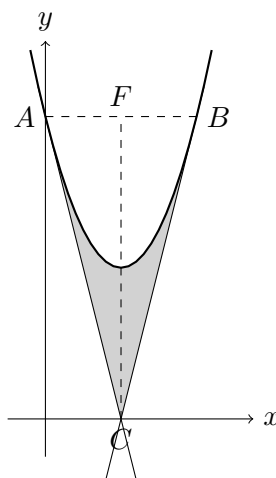
5. Tangent lines are drawn to the parabola $y = x^2 - 4x + 8$ at its points with x -coordinates 0 and 4.

a) What is the angle between the two tangent lines?

b) What is the area of the region bounded by the two tangent lines and the parabola?

(16 points)

Solution: a) Since $y' = 2x - 4$, the slopes of the tangent lines at $x = 0$ and $x = 4$ are -4 and 4 , and the points of tangency are $A(0, 8)$ and $B(4, 8)$. So the tangent lines are $e_1: y = -4x + 8$ and $e_2: y = 4x - 8$. Solving these two equations simultaneously, their intersection is $C(2, 0)$.



The angle between the two tangent lines is the angle ACB , the apex angle of the isosceles triangle ABC . In the right triangle ACF (F is the midpoint of AB), the angle at C is half of the angle in question. For this angle φ , $\tan \varphi = \frac{2}{8}$, so $\varphi \approx 14.036^\circ$. So the angle between the two tangent lines is $2\varphi \approx 28.1^\circ$.

b) We get the area T in question if from the area t under the parabola over $[0, 4]$ we subtract the area t_1 under e_1 over $[0, 2]$ and the area t_2 under e_2 over $[2, 4]$. Here $t_1 = t_2 = \frac{2 \cdot 8}{2} = 8$, and

$$t = \int_0^4 (x^2 - 4x + 8) dx = \left[\frac{x^3}{3} - 2x^2 + 8x \right]_0^4 = \frac{64}{3} - 32 + 32 = \frac{64}{3}$$

So the area in question is $T = t - (t_1 + t_2) = \frac{64}{3} - 16 = \frac{16}{3}$.

6. A pump can lift 4.8 m^3 of water per hour from a well. The water is let into a cylindrical pool whose top is a circular slab of area 154 m^2 . This slab also covers the 12 cm thick side wall of the pool.
- a) By how many centimeters does the water rise in the pool in 2.5 hours?
- b) The amount of water pumped in 15 minutes is filled into 25 liter and 60 liter containers. How many containers of each kind could there have been, if they were exactly filled with this amount of water?

(16 points)

Solution: a) From the area of the cover slab we get the radius R of the cylindrical pool (with its wall): $154 = R^2\pi$, so $R = \sqrt{\frac{154}{\pi}} \approx 7.00 \text{ m}$. Since the side wall is 12 cm thick, the inner radius of the cylinder is $r \approx 6.88 \text{ m}$.

In 2.5 hours the pump lifts $2.5 \cdot 4.8 = 12 \text{ m}^3$ of water. If this amount of water has height m in the cylindrical pool, then

$$12 = 6.88^2\pi \cdot m, \quad \text{so} \quad m = \frac{12}{6.88^2\pi} \approx 0.081$$

So in 2.5 hours the water level in the pool rises by only about 8.1 cm.

b) Since the pump lifts 4.8 m^3 of water per hour, in 15 minutes it lifts a quarter of this, 1200 liters. Let x be the number of 25 liter containers and y the number of 60 liter containers. Then

$$25x + 60y = 1200, \quad 5x + 12y = 240, \quad x = \frac{240 - 12y}{5} = 48 - \frac{12}{5}y$$

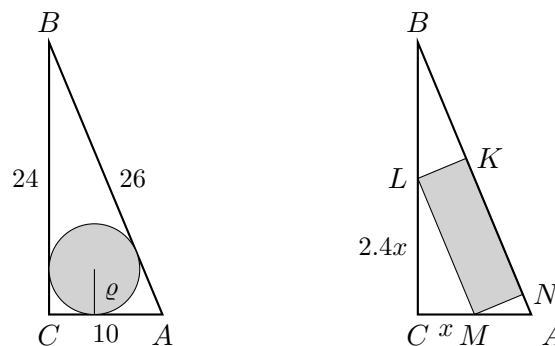
Since x and y are numbers of containers, y must be a natural number divisible by 5 for which x is also a natural number. The possible pairs of values are:

y (60 liter)	0	5	10	15	20
x (25 liter)	48	36	24	12	0

7. On two neighboring sides of a rectangular lot, we mark a point 10 meters and a point 24 meters from one of the vertices. We want to build a pool in the right triangle cut off by the line connecting the two points. According to one plan, we would build the largest circular pool; according to the other plan, we would build the largest rectangular pool with one full side on the hypotenuse of the right triangle. In which case, and by how much, would the water surface be larger?

(16 points)

Solution: The figure shows sketches of the two plans.



In the first case we need the incircle of the triangle. By the Pythagorean theorem, the third side is $c = \sqrt{24^2 + 10^2} = 26$. The area of the triangle is $t = \frac{10 \cdot 24}{2} = 120$. We can also use the formula $t = \rho s$, where ρ is the radius of the incircle and s is half the perimeter. So

$$s = \frac{10 + 24 + 26}{2} = 30, \quad \text{and} \quad \rho = \frac{t}{s} = \frac{120}{30} = 4$$

For the circular pool the water surface is $t_{\text{circle}} = 4^2\pi \approx 50.3 \text{ m}^2$.

In the second plan we look for the rectangle $KLMN$ of maximum area. Let $CM = x$; then $CL = 2.4x$, since LM is parallel to the hypotenuse. We write the area as a function of x . Let A' be the point of AB with LA' parallel to CA . The right triangles LBK and AMN are similar to ABC , with hypotenuses $LB = 2.4(10 - x)$ and $AM = 10 - x$; together they have the same area as the right triangle $A'BL$ with legs LB and $LA' = 10 - x$ (both are $1.2(10 - x)^2$). So

$$\begin{aligned} T(x) &= T_{ABC} - T_{MLC} - T_{LBK} - T_{AMN} = T_{ABC} - T_{MLC} - T_{A'BL} \\ &= \frac{10 \cdot 24}{2} - \frac{x \cdot 2.4x}{2} - \frac{(10 - x)(24 - 2.4x)}{2} = -2.4x(x - 10) \end{aligned}$$

This quadratic function has a negative leading coefficient and zeros 0 and 10, so it has its maximum at $x = 5$: $T(5) = -2.4 \cdot 5 \cdot (5 - 10) = 60$. So the rectangular pool would have a water surface of 60 m^2 .

So in this case we get a water surface larger by about 9.7 m^2 than with the circular plan.

8. The coordinates of three vertices of a hexagon are, in some order, $(-1, -1)$, $(2, 8)$, $(7, 3)$. Find the coordinates of the missing vertices if we know that each of the unknown points forms an isosceles trapezoid with the three given vertices.

(16 points)

Solution: The given points are $A(-1, -1)$, $B(2, 8)$, $C(7, 3)$. An isosceles trapezoid is a cyclic quadrilateral, so the points in question lie on the circumcircle of the triangle ABC . First we find the equation of this circle in the form $x^2 + y^2 + ax + by + c = 0$. Since A, B, C lie on it,

$$\begin{aligned} -a - b + c &= -2 \\ 2a + 8b + c &= -68 \\ 7a + 3b + c &= -58 \end{aligned}$$

Eliminating c : $3a + 9b = -66$ and $8a + 4b = -56$, that is, $a + 3b = -22$ and $2a + b = -14$. So $b = -6$, $a = -4$, $c = -12$, and the circle k is

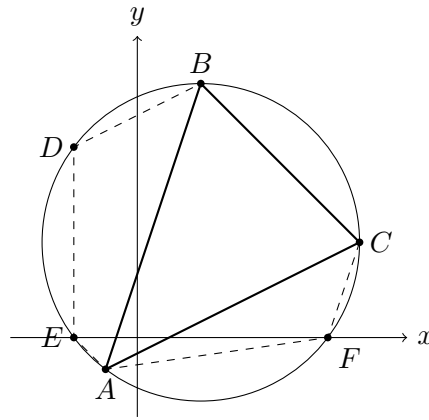
$$x^2 + y^2 - 4x - 6y - 12 = 0, \quad \text{that is,} \quad (x - 2)^2 + (y - 3)^2 = 25$$

There are three ways to place a fourth point so that it forms an isosceles trapezoid with A, B, C . Let us call these D, E, F .

Case I: D lies on the line e through B parallel to AC (and of course on the circle k). A direction vector of e is $\overrightarrow{AC}(8, 4)$, so a normal vector is $(1, -2)$, and the equation of e is $x - 2y = -14$. One solution of the system of the equations of k and e is of course the point B ; the other gives $D(-2, 6)$.

Case II: E lies on the line f through A parallel to BC . A direction vector of f is $\overrightarrow{BC}(5, -5)$, so a normal vector is $(1, 1)$, and the equation of f is $x + y = -2$. One solution of the system of the equations of k and f is the point A ; the other gives $E(-2, 0)$.

Case III: F lies on the line g through C parallel to AB . A direction vector of g is $\overrightarrow{AB}(3, 9)$, so a normal vector is $(3, -1)$, and the equation of g is $3x - y = 18$. One solution of the system of the equations of k and g is the point C ; the other gives $F(6, 0)$.



So the missing vertices of the hexagon are $(-2, 6)$, $(-2, 0)$, and $(6, 0)$.

9. Let

$$A = \frac{6n^2 - 15n - 9}{4n^2 - 18n - 10}$$

- What is the limit $\lim_{n \rightarrow \infty} A$?
- Find the integers n for which A is also an integer.
- Solve the equation $A = 3$.

(16 points)

Solution: a) Dividing the numerator and the denominator by n^2 and using the theorems on limits,

$$\lim_{n \rightarrow \infty} \frac{6n^2 - 15n - 9}{4n^2 - 18n - 10} = \lim_{n \rightarrow \infty} \frac{6 - \frac{15}{n} - \frac{9}{n^2}}{4 - \frac{18}{n} - \frac{10}{n^2}} = \frac{6}{4} = \frac{3}{2}$$

b) We find the zeros of the numerator and the denominator and factor the quadratic expressions. The roots of $6n^2 - 15n - 9 = 0$ are 3 and $-\frac{1}{2}$; the roots of $4n^2 - 18n - 10 = 0$ are 5 and $-\frac{1}{2}$. So $n \neq 5$ and $n \neq -\frac{1}{2}$, since otherwise the denominator would be zero, and

$$A = \frac{6(n-3)(n+\frac{1}{2})}{4(n-5)(n+\frac{1}{2})} = \frac{3(n-3)}{2(n-5)} = \frac{3(n-5)+6}{2(n-5)} = \frac{3}{2} + \frac{3}{n-5}$$

This is an integer exactly when $\frac{3}{n-5} = \frac{2A-3}{2}$ for an integer A , that is, when $n-5 = \frac{6}{2A-3}$. So $n-5$ must be 6 divided by an odd integer: $n-5 = \pm 6$ or ± 2 .

$n-5$	-6	-2	2	6
n	-1	3	7	11
A	1	0	3	2

So A is an integer exactly for $n = -1, 3, 7, 11$.

c) After factoring and simplifying as above,

$$\frac{3(n-3)}{2(n-5)} = 3, \quad 3n-9 = 6n-30, \quad n = 7$$

This is indeed a solution of the equation.

Note: KoMaL's solution works with the denominator $2n^2 - 9n - 5$, half of the one in the problem, so it gets the limit 3 and the integers $n = -1, 2, 3, 4, 6, 7, 8, 11$; these are not correct for the problem as stated. Part c) comes out the same.