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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. 520 students of a school took a test consisting of 9 questions. Each question could be answered yes or no. Everybody answered all the questions. The test was graded so that a correct answer to the k th question was worth k points, and a wrong answer $-k$ points.

- a) Show that there were certainly two students who filled out the test the same way.
b) At least how many students had the same score after grading?

(11 points)

2. a) Write 2012 as a sum of consecutive natural numbers.

- b) The sum of the first 2013 terms of an arithmetic sequence is 2012. Among the first 2012 terms, the sum of the terms with even index is one more than the sum of the terms with odd index. Find the first term of the sequence.

(13 points)

3. The function

$$I(x) = \int_{-2}^x f(t) dt$$

is defined on the real numbers, and we know that $f(t) = 3t^2 + 4t - 1$. Find the zeros of $I(x)$, and the intervals on which the function is convex or concave.

(13 points)

4. Take the base edge of the glass pyramid of the Louvre in Paris (a regular square pyramid) to be 34 m, and its height 22 m.

- a) How large a surface does the window-cleaning team have to clean, if they wash the glass panes both outside and inside?

At night the pyramid is to be lit by floodlights on posts. The posts stand at the vertices of a square the midpoints of whose sides are the vertices of the base of the pyramid.

- b) How high on the posts must the floodlights be placed, so that the light rays from them hit the glass surfaces perpendicularly at the centroids of the lateral faces?

(14 points)

Part II

5. The function $f(x) = \log_2(x+4) + 1$ is given. We know that the triangles A_nOC_n are isosceles right triangles with the right angle at O , where A_n is a point of the graph of $f(x)$ and O is the origin.
- Graph the function $f(x)$. For an arbitrary A_n , give the coordinates of the vertex C_n of the triangle.
 - Find the set of the points C_n , and graph it in the coordinate system.

(16 points)

6. Solve the following equations.

- $\frac{1}{\sqrt{2+x}} - \frac{1}{\sqrt{2-x}} = \sqrt{2}$
- $\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} = \sqrt{2}$

(16 points)

7. The vertex of the parabola p_2 is the point $C''(6, 7)$, its axis is parallel to that of the parabola $p_1: y = x^2 - 8x + 14$, and the point $P(9, \frac{19}{4})$ lies on the parabola p_2 .

The base of the isosceles triangles $A_nB_nC_n$ is A_nB_n , where $A_n \in p_2$, $B_n \in p_1$, and C_n lies on the y -axis of the coordinate system. The points A_n and B_n have the same first coordinate, and the second coordinate of B_n is less than that of A_n .

- Find the coordinates of the point C_1 , if the triangle $A_1B_1C_1$ is a right triangle.
- Which of the triangles $A_nB_nC_n$ has the largest area?

(16 points)

8. Find all polynomials $P(x) = ax^2 + bx + c$ with integer coefficients for which $|P(p_1)| = |P(p_2)| = |P(p_3)| = 5$, where p_1, p_2, p_3 are distinct primes.

(16 points)

9. At international sports events the competitors are driven not only by glory, but partly also by the large sums of money that come with advertising contracts. So it is not surprising that athletes and coaches sometimes use forbidden means to win.

To ensure the fairest possible conditions, doping tests are held according to set rules directly before important competitions, and also during the preparation. The athletes give urine samples, which are sealed, labeled, and kept and examined in two parts, the so-called A-sample and B-sample.

While the international sports federation works on developing a clear doping test that excludes all doubt, certain laboratories are developing doping substances that cannot be detected in the urine during the control. In part a), assume that at a sports event, 60 of the 2400 participants subjected to doping tests doped with a certain substance. All 2400 A-samples were checked.

The following holds for the test: if an athlete doped with this substance, the test detects it with 85% certainty; in this case we speak of a positive result. If an athlete did not use the substance, the test shows this with 96% certainty.

- Compute the probability that the A-sample gives a wrong verdict for a randomly chosen person.
- What is the probability that after the A-sample an athlete was accused without reason?
- Show that the chance that someone is accused without reason after the B-sample too is less than 0.08.

(16 points)