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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

- 520 students of a school took a test consisting of 9 questions. Each question could be answered yes or no. Everybody answered all the questions. The test was graded so that a correct answer to the k th question was worth k points, and a wrong answer $-k$ points.

- Show that there were certainly two students who filled out the test the same way.
- At least how many students had the same score after grading?

(11 points)

Solution: a) The test can be filled out in $2^9 = 512$ ways, so among 520 students there are certainly two who filled it out the same way.

b) The maximum score is $1 + 2 + \cdots + 9 = \frac{9 \cdot 10}{2} = 45$, and the minimum score is -45 . A wrong answer to a question lowers the score by an even number compared with the maximum, so the possible scores are the odd numbers between -45 and 45 . There are 46 odd numbers in this interval, so there can be 46 different final scores. (It is easy to see that each of them actually occurs.) If each score had been achieved by at most 11 students, then at most $46 \cdot 11 = 506$ students could attend the school.

So there were certainly at least 12 students with the same score.

- Write 2012 as a sum of consecutive natural numbers.
 - The sum of the first 2013 terms of an arithmetic sequence is 2012. Among the first 2012 terms, the sum of the terms with even index is one more than the sum of the terms with odd index. Find the first term of the sequence.

(13 points)

Solution: a) Let the smallest number be $k + 1$ and the largest $k + n$. Their sum is

$$nk + \frac{n(n+1)}{2} = 2012 = 2^2 \cdot 503, \quad \text{that is,} \quad n(2k + n + 1) = 8 \cdot 503$$

The two factors on the left side are natural numbers of different parity, and the second is greater than the first, so $n = 8$ (the case $n = 1$ is the trivial one-term sum). Since $2k + 8 + 1 = 503$, $k = 247$. So the consecutive natural numbers are

$$248 + 249 + 250 + 251 + 252 + 253 + 254 + 255 = 2012$$

- Since the sum of the first 2013 terms is 2012,

$$S_{2013} = 2013 \cdot \frac{2a_1 + 2012d}{2} = 2012, \quad \text{so} \quad a_1 = \frac{2012}{2013} - 1006d$$

Among the first 2012 terms, the sum of the terms with even index is one more than the sum of the terms with odd index:

$$(a_2 + a_4 + \cdots + a_{2012}) - (a_1 + a_3 + \cdots + a_{2011}) = (a_2 - a_1) + \cdots + (a_{2012} - a_{2011}) = 1006d = 1$$

So the first term of the sequence is $a_1 = \frac{2012}{2013} - 1 = -\frac{1}{2013}$.

3. The function

$$I(x) = \int_{-2}^x f(t) dt$$

is defined on the real numbers, and we know that $f(t) = 3t^2 + 4t - 1$. Find the zeros of $I(x)$, and the intervals on which the function is convex or concave.

(13 points)

Solution:

$$\begin{aligned} I(x) &= \int_{-2}^x (3t^2 + 4t - 1) dt = [t^3 + 2t^2 - t]_{-2}^x = x^3 + 2x^2 - x - (-8 + 8 + 2) \\ &= x^3 + 2x^2 - x - 2 = x^2(x + 2) - (x + 2) = (x + 2)(x + 1)(x - 1) \end{aligned}$$

So the zeros of $I(x)$ are -2 , -1 , and 1 .

The sign of the second derivative of I tells where it is convex or concave:

$$I''(x) = (3x^2 + 4x - 1)' = 6x + 4$$

So $I(x)$ is convex for $x > -\frac{2}{3}$ and concave for $x < -\frac{2}{3}$. (There is an inflection point at $x = -\frac{2}{3}$.)

4. Take the base edge of the glass pyramid of the Louvre in Paris (a regular square pyramid) to be 34 m, and its height 22 m.

a) How large a surface does the window-cleaning team have to clean, if they wash the glass panes both outside and inside?

At night the pyramid is to be lit by floodlights on posts. The posts stand at the vertices of a square the midpoints of whose sides are the vertices of the base of the pyramid.

b) How high on the posts must the floodlights be placed, so that the light rays from them hit the glass surfaces perpendicularly at the centroids of the lateral faces?

(14 points)

Solution: a) Let $ABCD$ be the base, E the apex, G the center of the base, and F the midpoint of BC . We know that $AB = 34$ and $EG = 22$. By the Pythagorean theorem, the slant height is

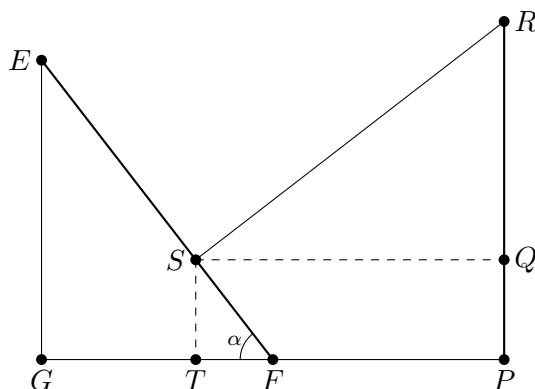
$$EF = \sqrt{22^2 + \left(\frac{34}{2}\right)^2} = \sqrt{773} \approx 27.80$$

We need twice the lateral surface of the pyramid:

$$2 \cdot 4 \cdot \frac{BC \cdot EF}{2} = 4 \cdot 34 \cdot \sqrt{773} \approx 3781 \text{ m}^2$$

So about 3781 m^2 of glass has to be washed.

b) The vertex P of the square of posts facing the side BC lies on the ray GF , with $GP = 34$ (the square of the posts has its vertices on the lines through G parallel to the sides of the base, at distance 34 from G), so $FP = 17$. We draw the important cross-section, the plane through E, G, F, P ; let S be the centroid of the lateral face BCE (on EF , with $SF = \frac{EF}{3}$), T its projection onto GP , R the floodlight, and Q the point of the post at the height of S .



In the right triangle GFE , $\tan \alpha = \frac{22}{17}$, so $\alpha \approx 52.3^\circ$, and $SF = \frac{EF}{3} \approx 9.27$. The ray SR is perpendicular to EF , so the angles GFE and QRS are acute angles with perpendicular sides, hence equal. Since $TF = \frac{17}{3} \approx 5.67$, we have $SQ = TF + FP \approx 5.67 + 17 = 22.67$. Then

$$\begin{aligned} QP &= ST = SF \cdot \sin \alpha \approx 9.27 \cdot \sin 52.3^\circ \approx 7.33 \\ \frac{SQ}{QR} &= \tan \alpha = \frac{22}{17}, \quad \text{so} \quad QR \approx 17.52 \\ PR &= QP + QR \approx 7.33 + 17.52 = 24.85 \end{aligned}$$

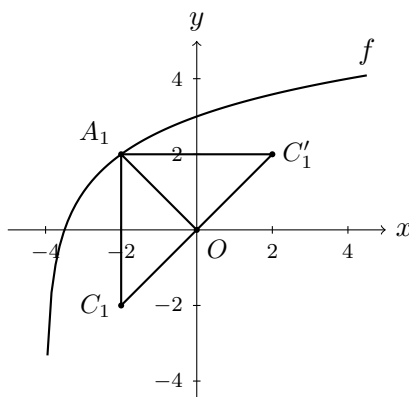
So the floodlights must be placed about 25 m high on the posts.

Part II

5. The function $f(x) = \log_2(x+4) + 1$ is given. We know that the triangles A_nOC_n are isosceles right triangles with the right angle at O , where A_n is a point of the graph of $f(x)$ and O is the origin.
- Graph the function $f(x)$. For an arbitrary A_n , give the coordinates of the vertex C_n of the triangle.
 - Find the set of the points C_n , and graph it in the coordinate system.

(16 points)

Solution: a) Choose an arbitrary point of the graph of $f(x)$, with first coordinate a . Its second coordinate is $f(a) = \log_2(a+4) + 1$, so the chosen point is $A_1(a, \log_2(a+4) + 1)$.



We get the third vertex of the triangle by rotating the chosen point A_1 through 90° about the origin. Depending on the direction of the rotation, there are two possibilities for the third vertex:

$$C_1(-\log_2(a+4) - 1, a) \quad \text{or} \quad C'_1(\log_2(a+4) + 1, -a)$$

Note: KoMaL writes $C_1(-\log_2(a+4)+1; -a)$ and $C'_1(\log_2(a+4)+1; a)$, which are not the rotated points (its figure shows the correct ones).

b) A rotation through 90° about O takes A_n to C_n . It can be replaced by the product of reflections in two lines through the origin forming an angle of 45° : $t_1: y = x$ and $t_2: y = 0$. So we must transform the graph of $f(x)$

1) by $t_1 \cdot t_2$, that is, by a rotation through $+90^\circ$ about O ;

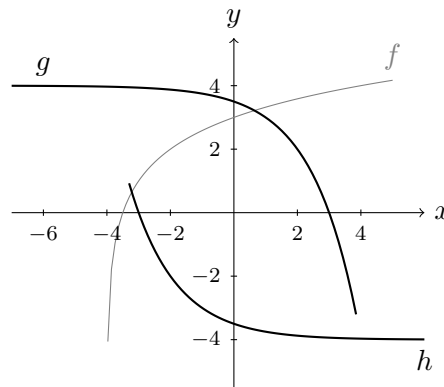
2) by $t_2 \cdot t_1$, that is, by a rotation through -90° about O .

1) Reflecting the set given by $y = \log_2(x+4) + 1$ in t_2 , we get $y = -\log_2(x+4) - 1$. Reflecting this in t_1 : $x = -\log_2(y+4) - 1$. Expressing y :

$$\log_2(y+4) = -(x+1), \quad y+4 = 2^{-(x+1)}, \quad y = \left(\frac{1}{2}\right)^{x+1} - 4$$

This is the graph of the function $h(x) = \left(\frac{1}{2}\right)^{x+1} - 4$.

2) Reflecting the set given by $y = \log_2(x+4) + 1$ in t_1 , we get $x = \log_2(y+4) + 1$, that is, $y = 2^{x-1} - 4$. Reflecting this in t_2 : $y = -2^{x-1} + 4$. This is the graph of the function $g(x) = -2^{x-1} + 4$.



So the set of the points C_n is the union of the graphs of g and h .

6. Solve the following equations.

a) $\frac{1}{\sqrt{2+x}} - \frac{1}{\sqrt{2-x}} = \sqrt{2}$

b) $\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} = \sqrt{2}$

(16 points)

Solution: a) The domain of the equation: $2+x > 0$ and $2-x > 0$, that is, $-2 < x < 2$. Multiplying both sides by $\sqrt{2+x} \cdot \sqrt{2-x}$:

$$\sqrt{2-x} - \sqrt{2+x} = \sqrt{2} \cdot \sqrt{4-x^2} = \sqrt{8-2x^2}$$

If $\sqrt{2-x} < \sqrt{2+x}$, then there is no solution, since the left side is negative and the right side is nonnegative. If $\sqrt{2-x} \geq \sqrt{2+x}$, that is, $-2 < x \leq 0$, we square both sides:

$$\begin{aligned} 4 - 2\sqrt{4-x^2} &= 2(4-x^2) \\ 0 &= \left(\sqrt{4-x^2}\right)^2 + \sqrt{4-x^2} - 2 \end{aligned}$$

By the quadratic formula, $\sqrt{4-x^2} = 1$ or $\sqrt{4-x^2} = -2$. The second is impossible; the first gives $x^2 = 3$. Of the two numbers only $x = -\sqrt{3}$ satisfies the conditions, so the original equation has only this root.

b) The domain of the equation: $\sin^2 x \neq 0$ and $\cos^2 x \neq 0$, that is, $x \neq k \cdot \frac{\pi}{2}$, $k \in \mathbb{Z}$. Multiplying both sides by $\sin^2 x \cdot \cos^2 x$:

$$\begin{aligned}\cos^2 x - \sin^2 x &= \sqrt{2} \sin^2 x \cos^2 x \\ \cos 2x &= \frac{\sqrt{2} (2 \sin x \cos x)^2}{4} = \frac{\sqrt{2} \sin^2 2x}{4} = \frac{\sin^2 2x}{2\sqrt{2}} \\ 2\sqrt{2} \cos 2x &= 1 - \cos^2 2x \\ \cos^2 2x + 2\sqrt{2} \cos 2x - 1 &= 0\end{aligned}$$

By the quadratic formula, $\cos 2x = -\sqrt{2} + \sqrt{3}$ or $\cos 2x = -\sqrt{2} - \sqrt{3}$. Since $\cos 2x \in [-1, 1]$, $\cos 2x = \sqrt{3} - \sqrt{2} \approx 0.3178$. So $2x \approx \pm 1.2473 + 2k\pi$, that is,

$$x \approx \pm 0.6237 + k\pi, \quad k \in \mathbb{Z}$$

7. The vertex of the parabola p_2 is the point $C''(6, 7)$, its axis is parallel to that of the parabola $p_1: y = x^2 - 8x + 14$, and the point $P(9, \frac{19}{4})$ lies on the parabola p_2 .

The base of the isosceles triangles $A_n B_n C_n$ is $A_n B_n$, where $A_n \in p_2$, $B_n \in p_1$, and C_n lies on the y -axis of the coordinate system. The points A_n and B_n have the same first coordinate, and the second coordinate of B_n is less than that of A_n .

a) Find the coordinates of the point C_1 , if the triangle $A_1 B_1 C_1$ is a right triangle.

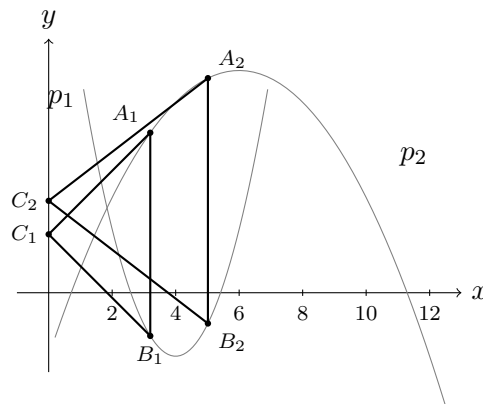
b) Which of the triangles $A_n B_n C_n$ has the largest area?

(16 points)

Solution: a) p_2 is a parabola opening downward: $(x - 6)^2 = -2p(y - 7)$. Since $P \in p_2$:

$$(9 - 6)^2 = -2p(4.75 - 7), \quad 9 = 4.5p, \quad p = 2$$

So the equation of p_2 is $(x - 6)^2 = -4(y - 7)$, that is, $y = -\frac{1}{4}(x - 6)^2 + 7 = -0.25x^2 + 3x - 2$.



The second coordinate of B_n is less than that of A_n , so

$$x^2 - 8x + 14 < -0.25x^2 + 3x - 2, \quad 1.25x^2 - 11x + 16 < 0, \quad 1.84 \approx x_1 < x < x_2 \approx 6.96$$

The isosceles triangle $A_1 B_1 C_1$ is a right triangle exactly when the difference of the second coordinates of A_1 and B_1 is twice their (common) first coordinate:

$$2x = (-0.25x^2 + 3x - 2) - (x^2 - 8x + 14) = -1.25x^2 + 11x - 16, \quad 0 = 1.25x^2 - 9x + 16$$

so $x = 3.2$ or $x = 4$. So there are two such triangles (both values are in the interval found above):

$$A_1(3.2, 5.04), B_1(3.2, -1.36), C_1(0, 1.84); \quad A'_1(4, 6), B'_1(4, -2), C'_1(0, 2)$$

Note: KoMaL lists $A_1(3.2; -1.36)$, $B_1(3.2; 5.04)$ and $A'_1(4; -2)$, $B'_1(4; 6)$, with the points A and B swapped; the points C are correct.

b) If the first coordinate of A_n and B_n is x , then the base of the triangle is $-1.25x^2 + 11x - 16$ and its height is x . So its area is

$$t(x) = \frac{x(-1.25x^2 + 11x - 16)}{2} = -0.625x^3 + 5.5x^2 - 8x$$

$t(x)$ has a maximum where its first derivative is 0 and changes from positive to negative:

$$t'(x) = -1.875x^2 + 11x - 8 = 0 \quad (1.84 < x < 6.96), \quad x_1 \approx 0.85 \quad \text{and} \quad x_2 \approx 5.016$$

$t'(x)$ changes from positive to negative at x_2 , so this is where $t(x)$ has its maximum. The maximum area is $t(5.016) \approx 19.376$, and the vertices (rounded) are $A_2(5.016, 6.758)$, $B_2(5.016, -0.968)$, $C_2(0, 2.895)$.

8. Find all polynomials $P(x) = ax^2 + bx + c$ with integer coefficients for which $|P(p_1)| = |P(p_2)| = |P(p_3)| = 5$, where p_1, p_2, p_3 are distinct primes.

(16 points)

Solution: We examine how many of the values $P(p_i)$ are negative. It is enough to discuss the cases where 0 or 1 of the values is negative, since in the other cases, replacing $P(x)$ by $-P(x)$, we get back to these cases.

Case 1: $P(p_1) = P(p_2) = P(p_3) = 5$. The polynomial $P(x) - 5$ has 3 distinct zeros, but its degree is at most 2, so $P(x) - 5$ is the constant polynomial.

Case 2: One of the values is negative, say $P(p_1) = -5$. Then $P(p_2) = P(p_3) = 5$, so p_2 and p_3 are zeros of $P(x) - 5$:

$$P(x) - 5 = a(x - p_2)(x - p_3)$$

Substituting p_1 : $-10 = a(p_1 - p_2)(p_1 - p_3)$. If all three primes were odd, the right side would be divisible by 4, but the left side is not. So one of the primes is 2.

i) If $p_1 = 2$, then $p_1 - p_2$ and $p_1 - p_3$ are negative odd numbers whose product is a positive odd number, so $a = -2$. The product of these two numbers is 5, so $p_1 - p_2 = -1$ and $p_1 - p_3 = -5$, that is, $p_2 = 3$ and $p_3 = 7$:

$$P_2(x) = -2(x - 3)(x - 7) + 5 = -2x^2 + 20x - 37$$

ii) If p_2 or p_3 is 2 (say $p_2 = 2$), then the factor $p_1 - p_3$ of the product on the right side is even, so its possible values are 2, -2, 10, -10, and $p_1 - 2$ is a positive integer:

$$\begin{array}{llll} p_1 - p_3 = 2: & -5 = a(p_1 - 2), & a = -1, p_1 = 7, p_3 = 5; & a = -5, p_1 = 3, p_3 = 1 \\ p_1 - p_3 = -2: & 5 = a(p_1 - 2), & a = 1, p_1 = 7, p_3 = 9; & a = 5, p_1 = 3, p_3 = 5 \\ p_1 - p_3 = 10: & -1 = a(p_1 - 2), & a = -1, p_1 = 3, p_3 = -7 \\ p_1 - p_3 = -10: & 1 = a(p_1 - 2), & a = 1, p_1 = 3, p_3 = 13 \end{array}$$

The conditions of the problem are satisfied by $(p_1, p_2, p_3) = (7, 2, 5), (3, 2, 5), (3, 2, 13)$. The polynomials obtained are

$$\begin{aligned} P_3(x) &= -(x - 2)(x - 5) + 5 = -x^2 + 7x - 5 \\ P_4(x) &= 5(x - 2)(x - 5) + 5 = 5x^2 - 35x + 55 \\ P_5(x) &= (x - 2)(x - 13) + 5 = x^2 - 15x + 31 \end{aligned}$$

So 10 polynomials satisfy the conditions of the problem: the 5 listed above and their negatives.

9. At international sports events the competitors are driven not only by glory, but partly also by the large sums of money that come with advertising contracts. So it is not surprising that athletes and coaches sometimes use forbidden means to win.

To ensure the fairest possible conditions, doping tests are held according to set rules directly before important competitions, and also during the preparation. The athletes give urine samples, which are sealed, labeled, and kept and examined in two parts, the so-called A-sample and B-sample.

While the international sports federation works on developing a clear doping test that excludes all doubt, certain laboratories are developing doping substances that cannot be detected in the urine during the control. In part a), assume that at a sports event, 60 of the 2400 participants subjected to doping tests doped with a certain substance. All 2400 A-samples were checked.

The following holds for the test: if an athlete doped with this substance, the test detects it with 85% certainty; in this case we speak of a positive result. If an athlete did not use the substance, the test shows this with 96% certainty.

- Compute the probability that the A-sample gives a wrong verdict for a randomly chosen person.
- What is the probability that after the A-sample an athlete was accused without reason?
- Show that the chance that someone is accused without reason after the B-sample too is less than 0.08.

(16 points)

Solution: a) The result is wrong if the test of a doped athlete is negative, or the test of an athlete who did not dope is positive. Let D be the event that a competitor doped, and T the event that the result of the doping test is positive. We are given

$$P(D) = \frac{60}{2400} = 0.025, \quad P(T | D) = 0.85, \quad P(\bar{T} | \bar{D}) = 0.96$$

So

$$P(\bar{T}D) + P(T\bar{D}) = (1 - 0.85) \cdot 0.025 + (1 - 0.96)(1 - 0.025) = 0.04275$$

So for a randomly chosen person the result is wrong with probability about 4.3%.

- b) We need $P(\bar{D} | T)$:

$$P(\bar{D} | T) = \frac{P(T | \bar{D}) P(\bar{D})}{P(T | D) P(D) + P(T | \bar{D}) P(\bar{D})} = \frac{0.04 \cdot 0.975}{0.85 \cdot 0.025 + 0.04 \cdot 0.975} = \frac{0.039}{0.02125 + 0.039} \approx 0.6473$$

So the probability that an athlete accused of doping did not dope is about 65%. The denominator shows that most of the athletes accused of doping did not dope with this (unknown) substance: the large number of athletes who did not dope produces many wrong verdicts from the error of the test.

- c) The B-sample is examined only if the result of the A-sample is positive. In this case, by part b), the probabilities that someone did not dope or did dope are $P(\bar{D}_B) = 0.6473$ and $P(D_B) = 1 - 0.6473 = 0.3527$. As in part b),

$$P(\bar{D}_B | T_2) = \frac{0.04 \cdot 0.6473}{0.85 \cdot 0.3527 + 0.04 \cdot 0.6473} \approx 0.0795 < 0.08$$