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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

- Which three natural numbers are consecutive terms of an arithmetic sequence, if the sum of their squares is 264, and their product equals the quotient of 1792 and the middle number?

(10 points)

Solution: Three consecutive terms of an arithmetic sequence are $b - d$, b , $b + d$, where b is the middle term and d is the common difference. By the problem,

$$\begin{aligned}(b - d)^2 + b^2 + (b + d)^2 &= 3b^2 + 2d^2 = 264 \\ (b - d)b(b + d) &= b(b^2 - d^2) = \frac{1792}{b}, \quad \text{that is, } b^4 - b^2d^2 = 1792\end{aligned}$$

From the first equation $d^2 = 132 - \frac{3}{2}b^2$; substituting into the second, $b^4 - b^2(132 - \frac{3}{2}b^2) = 1792$, which leads to a quadratic equation in b^2 :

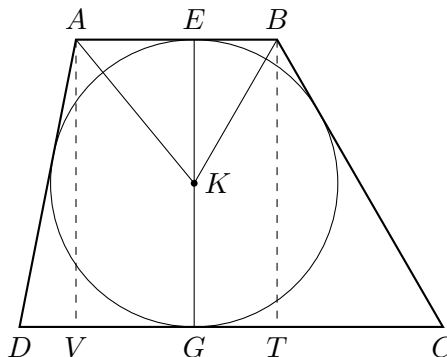
$$5b^4 - 264b^2 - 3584 = 0$$

One of its roots is negative, so it cannot be a square; the other is $b^2 = 64$, so $b = 8$ ($b = -8$ is impossible, since b is a natural number). Substituting back into the first equation, $d = 6$. So the three natural numbers are 2, 8, and 14.

- What is the area of the tangential trapezoid one of whose legs forms an angle of 120° with the 7 cm long parallel side, and whose inscribed circle has radius 5 cm?

(13 points)

Solution: The figure shows the trapezoid to scale, with $AB = 7$, $KE = KF = KG = KH = 5$, and $\angle ABC = 120^\circ$.



The height of the trapezoid is the diameter of the inscribed circle: $AV = EG = BT = 10$. To find the area we need the length of DC . Since $\angle TCB = 60^\circ$, in the right triangle TCB we have $BT = \sqrt{3} \cdot TC$, so $TC = \frac{10}{\sqrt{3}} \approx 5.77$.

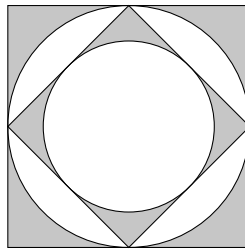
We find DV from the right triangle ADV , for which we need one more angle. In the right triangle KEB the angle at B is 60° (BK bisects the angle ABC), so $KE = \sqrt{3} \cdot EB$, and $EB = \frac{5}{\sqrt{3}} \approx 2.89$; so $AE = 7 - 2.89 = 4.11$. Then in the right triangle AEK , $\tan \angle EAK = \frac{5}{4.11}$, so $\angle EAK \approx 50.56^\circ$. Since AK bisects the angle DAB ,

$$\angle DAV = 2 \cdot \angle EAK - 90^\circ \approx 11.12^\circ, \quad \text{so} \quad DV = AV \tan 11.12^\circ \approx 1.97$$

So $DC = DV + VT + TC \approx 1.97 + 7 + 5.77 = 14.74$, and the area of the trapezoid is

$$T = \frac{AB + DC}{2} \cdot AV \approx \frac{7 + 14.74}{2} \cdot 10 \approx 108.7 \text{ cm}^2$$

3. The figure shows a square with side 1 unit, its inscribed circle, the square determined by the midpoints of its sides, and the inscribed circle of that square too.
- What percent of the large square was colored gray?
 - We repeat this procedure infinitely many times. What percent of the large square is colored gray this way?



(14 points)

Solution: a) The gray area is the sum of the difference of the areas of the large square and the large circle, and of the small square and the small circle:

$$T_{\text{gray}} = 1^2 - \left(\frac{1}{2}\right)^2 \pi + \left(\sqrt{\frac{1}{2}}\right)^2 - \left(\frac{\sqrt{2}}{4}\right)^2 \pi = 1 - \frac{1}{4}\pi + \frac{1}{2} - \frac{1}{8}\pi \approx 0.322$$

So about 32.2% of the square is gray.

b) The sides of the nested squares and the radii of the circles form geometric sequences, so the gray area is

$$T_{\text{gray}} = 1 - \frac{1}{4}\pi + \frac{1}{2} - \frac{1}{8}\pi + \frac{1}{4} - \frac{1}{16}\pi + \cdots = \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) - \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) \pi$$

Summing the two geometric series:

$$T_{\text{gray}} = \frac{1}{1 - \frac{1}{2}} - \frac{\frac{1}{4}}{1 - \frac{1}{2}} \pi = 2 - \frac{1}{2}\pi \approx 0.429$$

So in this case about 42.9% of the square is gray.

Remark: The problem can also be solved without geometric series.

4. Find the two consecutive integers between which the following sum lies.

$$\lg \left(11 \cdot \sqrt{\frac{1}{3}} \right) + \lg \left(11^2 \cdot \sqrt{\frac{3}{5}} \right) + \lg \left(11^3 \cdot \sqrt{\frac{5}{7}} \right) + \cdots + \lg \left(11^{500} \cdot \sqrt{\frac{999}{1001}} \right)$$

(14 points)

Solution: Using the identity for the logarithm of a product and regrouping the terms:

$$(\lg 11 + \lg 11^2 + \cdots + \lg 11^{500}) + \left(\lg \sqrt{\frac{1}{3}} + \lg \sqrt{\frac{3}{5}} + \cdots + \lg \sqrt{\frac{999}{1001}} \right)$$

By the identity for the logarithm of a power, the first sum is

$$(1 + 2 + 3 + \cdots + 500) \lg 11 = \frac{500 \cdot 501}{2} \lg 11 = 125\,250 \lg 11$$

The second sum telescopes:

$$\lg \sqrt{\frac{1}{3} \cdot \frac{3}{5} \cdot \cdots \cdot \frac{999}{1001}} = \lg \sqrt{\frac{1}{1001}}$$

So the sum is $125\,250 \lg 11 + \lg \sqrt{\frac{1}{1001}} \approx 130\,432.93$, and the two consecutive integers are 130 432 and 130 433.

Part II

5. Spherical chocolates of radius 2 cm are packed five at a time into boxes shaped like square-based pyramids, so that four are at the bottom and one on top. Each lower sphere touches the upper one and two neighboring lower ones, and also touches the base of the pyramid and two neighboring lateral faces. The upper sphere touches all four lateral faces.

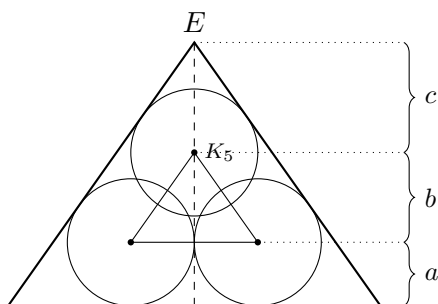
a) How tall is the box?

b) What is the surface area of the box?

(16 points)

Solution: a) The centers K_1, K_2, K_3, K_4, K_5 of the spheres form a small square-based pyramid all of whose edges are 4 cm (the spheres touch each other). Each lateral face of the box contains two lower centers and the upper one at distance 2 cm, so the faces of the small pyramid are parallel to those of the box: the box is similar to the small pyramid, so all of its edges are equal too, and the two pyramids have a common axis.

The figure shows the projection of the box onto a plane through the axis, parallel to two opposite base edges. The two faces perpendicular to this plane appear as the slanted sides, and the spheres appear as circles of radius 2.



The height of the box consists of three parts. The lowest part is $a = 2$, the distance of the lower centers from the base. The middle part is the height of the small pyramid. By the Pythagorean theorem in the right triangle formed by this height, a lateral edge, and half a diagonal of the base,

$$b^2 = 4^2 - \left(\frac{\sqrt{2}}{2} \cdot 4\right)^2 = 8, \quad \text{so} \quad b = \sqrt{8} \approx 2.83$$

For the top part we use the right triangle whose vertices are the apex E , the center K_5 , and the point where the upper sphere touches a lateral face: its hypotenuse is c and one leg is 2, so $\sin \alpha = \frac{2}{c}$, where α is the angle between the axis and the lateral face. The same angle appears in the small pyramid, between its height and the altitude of a lateral face (whose length is $\frac{\sqrt{3}}{2} \cdot 4$, since the faces are equilateral triangles):

$$\sin \alpha = \frac{2}{\frac{\sqrt{3}}{2} \cdot 4} = \frac{1}{\sqrt{3}}, \quad \text{so} \quad c = \frac{2}{\sin \alpha} = 2\sqrt{3} \approx 3.46$$

So the height of the box is $m = a + b + c = 2 + 2\sqrt{2} + 2\sqrt{3} \approx 8.29$ cm.

b) For the surface area we need the edge length x of the box. In the right triangle formed by the height, a lateral edge, and half a diagonal of the base,

$$m^2 = x^2 - \left(\frac{\sqrt{2}}{2}x\right)^2 = \frac{x^2}{2}, \quad \text{so} \quad x = \sqrt{2}m \approx 11.727$$

The faces are a square and four equilateral triangles, so

$$A_{\text{box}} = x^2 + 4 \cdot \frac{\sqrt{3}}{4}x^2 = (1 + \sqrt{3})x^2 \approx 375.7 \text{ cm}^2$$

Note: KoMaL uses the rounded value $x \approx 11.72$ and gets 375.27 cm^2 .

6. Solve the following equations over the real numbers.

a) $\sqrt{x} + \sqrt{2x+1} = \sqrt{3x+13}$

b) $\frac{\log_2(x^3 + 26)}{\log_4(x+2)} = 6$

c) $2\sin^2 x + \sin^2 2x = 2$

(16 points)

Solution: a) The expressions under the square roots cannot be negative, so $x \geq 0$. Squaring both sides,

$$x + 2x + 1 + 2\sqrt{x}\sqrt{2x+1} = 3x + 13, \quad \text{that is,} \quad \sqrt{x(2x+1)} = 6$$

Squaring again, $x(2x+1) = 36$, so $2x^2 + x - 36 = 0$. Its roots are 4 and $-\frac{9}{2}$, and the second does not satisfy the condition. So the solution is $x = 4$, which we also checked.

b) The arguments of the logarithms must be positive: $x^3 + 26 > 0$ and $x + 2 > 0$; and the denominator cannot be zero: $x \neq -1$. Altogether: $x > -2$, $x \neq -1$. Since $\log_2 a = \frac{\log_4 a}{\log_4 2} = 2 \log_4 a$,

$$2 \log_4(x^3 + 26) = 6 \log_4(x + 2)$$

$$\log_4(x^3 + 26) = \log_4(x + 2)^3$$

$$x^3 + 26 = (x + 2)^3$$

$$0 = 6x^2 + 12x - 18, \quad \text{that is,} \quad x^2 + 2x - 3 = 0$$

The roots are 1 and -3 ; because of the conditions, only $x = 1$ is a solution.

c) Using $\sin 2x = 2 \sin x \cos x$ and $\cos^2 x = 1 - \sin^2 x$:

$$\begin{aligned} 2 \sin^2 x + 4 \sin^2 x (1 - \sin^2 x) &= 2 \\ 0 &= 2 \sin^4 x - 3 \sin^2 x + 1 \end{aligned}$$

The roots of this quadratic equation in $\sin^2 x$ are 1 and $\frac{1}{2}$. So $\sin x = \pm 1$ or $\sin x = \pm \frac{\sqrt{2}}{2}$, that is,

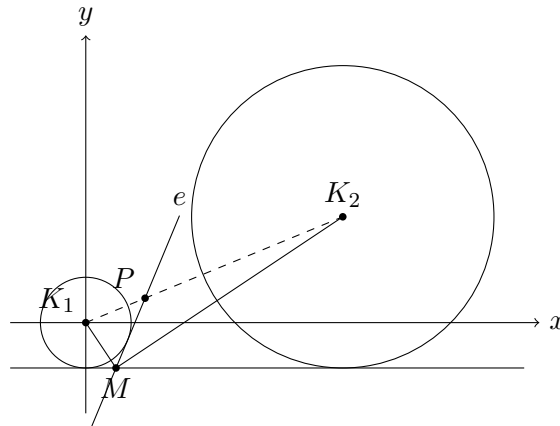
$$x = \frac{\pi}{2} + l\pi \quad \text{or} \quad x = \frac{\pi}{4} + m \cdot \frac{\pi}{2}, \quad l, m \in \mathbb{Z}$$

Altogether: $x = \frac{k\pi}{4}$, where k is an integer not divisible by 4.

7. In the coordinate system the circles $k_1: x^2 + y^2 = 9$ and $k_2: (x - 17)^2 + (y - 7)^2 = 100$ are given. Prove that the segment joining the centers of the two circles subtends a right angle at the intersection of the horizontal common external tangent and the common internal tangent with positive slope of the two circles.

(16 points)

Solution: The centers are $K_1(0, 0)$ and $K_2(17, 7)$, with radii 3 and 10. The horizontal common external tangent is $y = -3$, since both circles reach down to $y = -3$.



The internal common tangent passes through the internal center of similarity P of the two circles, which divides the segment K_1K_2 in the ratio 3 : 10:

$$P \left(17 \cdot \frac{3}{13}, 7 \cdot \frac{3}{13} \right) = \left(\frac{51}{13}, \frac{21}{13} \right)$$

A line through P with slope m is $y - \frac{21}{13} = m \left(x - \frac{51}{13} \right)$. It is tangent to k_1 if its distance from K_1 is 3:

$$\frac{|21 - 51m|}{13\sqrt{1 + m^2}} = 3, \quad (21 - 51m)^2 = 1521(1 + m^2), \quad 60m^2 - 119m - 60 = 0$$

so $m = \frac{12}{5}$ or $m = -\frac{5}{12}$. The tangent with positive slope is $y - \frac{21}{13} = \frac{12}{5} \left(x - \frac{51}{13} \right)$, that is, $e: 12x - 5y = 39$. (Its distance from K_2 is $\frac{|204 - 35 - 39|}{13} = 10$, so it is indeed tangent to k_2 too.)

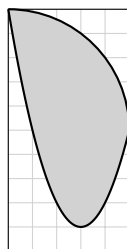
The intersection M of the two tangents is the solution of the system $y = -3$, $12x - 5y = 39$: $M(2, -3)$. Finally, $\overrightarrow{MK_1} = (-2, 3)$ and $\overrightarrow{MK_2} = (15, 10)$, so

$$\overrightarrow{MK_1} \cdot \overrightarrow{MK_2} = -30 + 30 = 0$$

So the segment K_1K_2 indeed subtends a right angle at M .

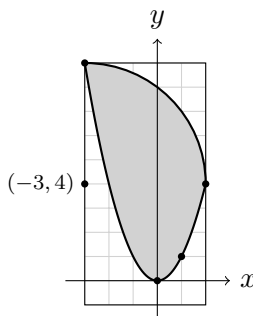
Remark: This is true in general: the segment joining the centers of two circles always subtends a right angle at the intersection of a common external and a common internal tangent.

8. A tailor uses the pattern shown in the figure for a carnival butterfly badge, and cuts it out of a rectangular piece of fabric bought in the store. The pattern consists of an arc of a parabola and an arc of a circle.
- What percent of the fabric will be waste?
 - Find all the intersection points of the parabola and the circle.



(16 points)

Solution: a) Place the pattern in a coordinate system as in the figure. Then the parabola is the graph of $f(x) = x^2$, the fabric is the rectangle $-3 \leq x \leq 2$, $-1 \leq y \leq 9$, and the circle has center $(-3, 4)$ and radius 5.



The area under the parabola (down to the x -axis) is

$$\int_{-3}^2 x^2 dx = \left[\frac{x^3}{3} \right]_{-3}^2 = \frac{8}{3} - \frac{-27}{3} = \frac{35}{3}$$

The part below the x -axis has area 5, so the waste here is $\frac{35}{3} + 5 = \frac{50}{3} \approx 16.667$. In the upper right part, a quarter circle of radius 5 is missing from a 5×5 square, so this waste has area $25 - \frac{25\pi}{4} \approx 5.365$. So the total waste is about 22.032. The area of the original rectangle is 50, so about 44.1% of the fabric is waste.

b) The equation of the parabola is $y = x^2$, and the equation of the circle is $(x + 3)^2 + (y - 4)^2 = 25$. Substituting $y = x^2$ into the second equation:

$$\begin{aligned} x^2 + 6x + 9 + x^4 - 8x^2 + 16 &= 25 \\ x^4 - 7x^2 + 6x &= 0 \\ x(x^3 - 7x + 6) &= 0 \end{aligned}$$

Since the pattern already shows two intersection points (at $x = -3$ and $x = 2$), the factored form is easy to write: $x(x + 3)(x - 2)(x - 1) = 0$. Substituting back, we get the four intersection points: $(-3, 9)$, $(0, 0)$, $(1, 1)$, $(2, 4)$.

9. At the 2012 London Olympics the Hungarian team won 8 gold, 4 silver, and 5 bronze medals. There was no event in which two Hungarian medals were won.
- In how many ways can we list the events in which our team won a medal, so that first the events of the gold medals, then those of the silver medals, and finally those of the bronze medals are named?
 - In how many ways can we list the above 17 events, if among the events of medals of the same color we always list first the one that ended earlier?
 - What is the probability that choosing three of our medal-winning events at random, the corresponding medals will be of different colors?

(16 points)

Solution: a) We need the product of the numbers of permutations of the 8 gold, 4 silver, and 5 bronze medal events: $8! \cdot 4! \cdot 5! = 116\,121\,600$.

b) We must merge three sequences of lengths 8, 4, and 5, each in a fixed order. Among the 8 gold medal events there are 9 places where the silver medal events can go, possibly several in the same place; this is a combination with repetition, possible in $\binom{12}{4}$ ways. In the resulting sequence of length 12 there are 13 places for the bronze medal events, which gives $\binom{17}{5}$ possibilities. So altogether there are

$$\binom{12}{4} \cdot \binom{17}{5} = 3\,063\,060$$

lists. (This is also $\frac{17!}{8! \cdot 4! \cdot 5!}$.)

c) We choose three of the 17 events, so the number of all cases is $\binom{17}{3} = 680$. One gold, one silver, and one bronze medal event can be chosen in $8 \cdot 4 \cdot 5 = 160$ ways. So the probability in question is

$$P = \frac{160}{680} = \frac{4}{17} \approx 0.235$$