

author: László Számadó

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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The functions f and g are defined on the set of real numbers:

$$\begin{aligned}f(x) &= (2x - 1)^2 + (x - 4)(x + 4) - 5x(x - 1) \\g(x) &= x^3 - 4|x|\end{aligned}$$

- a) Prove that f is a linear function.
b) Find the zeros of the function g .

(11 points)

2. Five of the sides and diagonals of a regular hexagon are painted red, and the others green. Then we choose five of these segments at random. What is the probability that exactly three red and two green segments are chosen?

(13 points)

3. ABC is an equilateral triangle. On the smaller arc BC of the circle with center A and radius AB , let D be the point of trisection closer to B and E the point of trisection closer to C . Let F be the midpoint of the smaller arc AC of the circle with center B and radius AB . Find the interior angles of the triangle formed by the lines AD , AE , and BF .

(13 points)

4. The coordinates of the vertices of the triangle ABC are $A(1, 1)$, $B(6, 2)$, and $C(2, 6)$.

- a) How long is the shortest altitude of the triangle?
b) What is the area of the triangle?
c) We roll a red and a green die at the same time. The number rolled with the red die is the first coordinate of a point, and the number rolled with the green die is its second coordinate. What is the probability that the point obtained this way is in the interior of the triangle ABC ?

(14 points)

Part II

5. At a class reunion in 2012, Laszlo's former students asked him how old he was. He answered: "My mother's year of birth is $\overline{abcd}$, and my year of birth is $\overline{ab}^2 + \overline{cd}^2$; she was 21 years old then. We do not live in the same town; I am going to visit her next week."

How old is Laszlo in 2012?

(16 points)

6. Consider the sequence $\{a_n\} = \left\{ \frac{2n+12}{n+3} \right\}$ ($n \in \mathbb{N}^+$).

- a) Find all terms of the sequence that are not less than 3.
- b) The indices of a_1, a_3, a_9 are three consecutive terms of a geometric sequence. Prove that these three terms of the sequence are three consecutive terms of an arithmetic sequence.
- c) How many terms of the sequence have the value 2.012 when rounded to three decimal places?
- d) Find $\lim_{n \rightarrow \infty} a_n$.

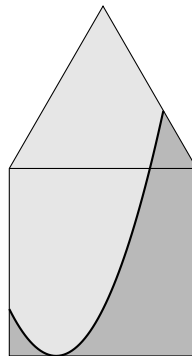
(16 points)

7. In the cuboid $ABCDEFGH$ the vertices are labeled so that on the face congruent to the base $ABCD$ the vertex E is joined by an edge to A , F to B , G to C , and H to D . We know that $\angle BAF = 45^\circ$ and $\angle CAG = 30^\circ$.

- a) Prove that $AFGD$ is a square.
- b) What is the surface area of the tetrahedron $AFBC$, if $AB = a$?
- c) What is the volume of the tetrahedron $AFHC$, if the distance between AF and HC is $a\sqrt{2}$?

(16 points)

8. The gable wall of a house consists of a square and an equilateral triangle. The wall is to be plastered in two colors. The boundary line between the two parts will be a parabola, as shown in the figure. The part of the wall above the parabola will be plastered in a lighter color, and the rest in a darker color. What percent of the surface will be the darker shade?



(16 points)

9. Find the real numbers x for which the sum of the squares of $\cos x$ and $\cos 2x$ equals the square of $\cos 3x$.

(16 points)