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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The functions f and g are defined on the set of real numbers:

$$\begin{aligned} f(x) &= (2x-1)^2 + (x-4)(x+4) - 5x(x-1) \\ g(x) &= x^3 - 4|x| \end{aligned}$$

- a) Prove that f is a linear function.
b) Find the zeros of the function g .

(11 points)

Solution: a) The formula defining f can be simplified:

$$(2x-1)^2 + (x-4)(x+4) - 5x(x-1) = 4x^2 - 4x + 1 + x^2 - 16 - 5x^2 + 5x = x - 15$$

So f is indeed a linear function.

b) We need the solutions of the equation $x^3 - 4|x| = 0$. If $x \geq 0$, the equation becomes $x^3 - 4x = 0$, that is, $x(x-2)(x+2) = 0$; because of the condition, $x = 0$ and $x = 2$ are the solutions here. If $x < 0$, the equation is $x^3 + 4x = 0$, that is, $x(x^2 + 4) = 0$. The second factor is positive for every real x , and the first is zero only for $x = 0$, so under this condition there is no solution.

So the function g has two zeros, 0 and 2.

2. Five of the sides and diagonals of a regular hexagon are painted red, and the others green. Then we choose five of these segments at random. What is the probability that exactly three red and two green segments are chosen?

(13 points)

Solution: A regular hexagon has 6 sides and 9 diagonals. Of the 15 segments, 5 are red and 10 are green. First we find the number of all possible choices: we choose five of the 15 segments (the order does not matter), which can be done in $\binom{15}{5} = 3003$ ways. Then we count the favorable cases: the three red segments are chosen from the five red ones and the two green ones from the ten green ones (the order does not matter here either), which can be done in

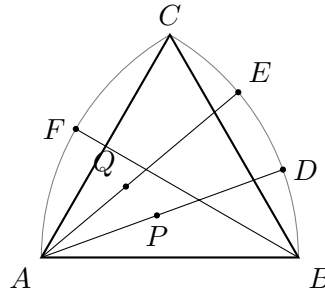
$$\binom{5}{3} \binom{10}{2} = 10 \cdot 45 = 450$$

ways. The probability in question is $p = \frac{450}{3003} \approx 0.1499$.

3. ABC is an equilateral triangle. On the smaller arc BC of the circle with center A and radius AB , let D be the point of trisection closer to B and E the point of trisection closer to C . Let F be the midpoint of the smaller arc AC of the circle with center B and radius AB . Find the interior angles of the triangle formed by the lines AD , AE , and BF .

(13 points)

Solution: Let P be the intersection of AD and BF , and Q the intersection of AE and BF . We must find the interior angles of the triangle APQ .



Each angle of the equilateral triangle ABC is 60° . By the inscribed angle theorem (here the arcs are about A , so the central angle BAC is divided equally), AD and AE divide the 60° angle at the vertex A into three equal parts. So in the triangle AQP the angle at A is 20° , and in the triangle BAP the angle at A is also 20° . The line BF bisects the 60° angle at the vertex B , so $\angle ABP = 30^\circ$.

The exterior angle of the triangle ABP at P is the interior angle of the triangle APQ at P , and it equals the sum of the two other interior angles of the triangle ABP : $\angle APQ = 20^\circ + 30^\circ = 50^\circ$. Knowing two interior angles, the third is $\angle AQP = 180^\circ - 20^\circ - 50^\circ = 110^\circ$.

So the angles of the triangle in question are 20° , 50° , and 110° .

4. The coordinates of the vertices of the triangle ABC are $A(1, 1)$, $B(6, 2)$, and $C(2, 6)$.
- How long is the shortest altitude of the triangle?
 - What is the area of the triangle?
 - We roll a red and a green die at the same time. The number rolled with the red die is the first coordinate of a point, and the number rolled with the green die is its second coordinate. What is the probability that the point obtained this way is in the interior of the triangle ABC ?

(14 points)

Solution: a) The shortest altitude belongs to the longest side. We find the lengths of the sides by the distance formula:

$$\begin{aligned} AB &= \sqrt{(1-6)^2 + (1-2)^2} = \sqrt{26} \\ BC &= \sqrt{(6-2)^2 + (2-6)^2} = \sqrt{32} = 4\sqrt{2} \\ CA &= \sqrt{(2-1)^2 + (6-1)^2} = \sqrt{26} \end{aligned}$$

So the altitude in question is the distance of the vertex A from the line BC . By our calculation ABC is an isosceles triangle, since $AB = AC$. The altitude in question belongs to the base, so its foot is the midpoint of the base, $T(4, 4)$, and its length is

$$TA = \sqrt{(4-1)^2 + (4-1)^2} = 3\sqrt{2}$$

- b) From $BC = 4\sqrt{2}$ and $TA = 3\sqrt{2}$, the area is

$$T_{ABC} = \frac{4\sqrt{2} \cdot 3\sqrt{2}}{2} = 12$$

Remark: The area can also be found directly, without part a): from the square ADEF (where $D(6, 1)$, $E(6, 6)$, $F(1, 6)$) we cut off the right triangles ADB , AFC , and BCE : $T_{ABC} = 25 - 2.5 - 2.5 - 8 = 12$.

c) Among the vertices of the triangle, A has the smallest first coordinate and B the largest. So the points inside the triangle have first coordinate 2, 3, 4, or 5. For these cases we can quickly find the possible second coordinates:

$$(2, 2), (2, 3), (2, 4), (2, 5), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (5, 2)$$

Each of these can be the result of rolling the dice, so the number of favorable cases is 10. The number of all cases (the number of points that can be obtained by rolling the two dice) is $6 \cdot 6 = 36$. So the probability in question is $\frac{10}{36} = \frac{5}{18} \approx 0.278$.

Part II

5. At a class reunion in 2012, Laszlo's former students asked him how old he was. He answered: "My mother's year of birth is $\overline{abcd}$, and my year of birth is $\overline{ab}^2 + \overline{cd}^2$; she was 21 years old then. We do not live in the same town; I am going to visit her next week."

How old is Laszlo in 2012?

(16 points)

Solution: Since in 2012 Laszlo was talking with students who had graduated earlier, he was born in the 20th century. He is visiting his mother next week, so she was certainly also born in the 20th century. (If she lived to be more than 112 years old, we would surely have heard about it in the media: the oldest person in Hungary was 110 years old on April 27, 2012.) So only $a = 1$, $b = 9$ are possible. By the text,

$$\begin{aligned}\overline{abcd} + 21 &= \overline{ab}^2 + \overline{cd}^2 \\ 1900 + 10c + d + 21 &= 19^2 + 100c^2 + 20cd + d^2 \\ 1560 + 10c - 100c^2 - 20cd &= d^2 - d\end{aligned}$$

The left side is divisible by 10, so the right side, $d(d - 1)$, must also be divisible by 10. So the possible values of the digit d are 6, 5, 1, 0. Each case gives a quadratic equation for c , and we need the cases where it has a digit as a solution:

Case I: $d = 6$: $10c^2 + 11c - 153 = 0$; no digit c .

Case II: $d = 5$: $10c^2 + 9c - 154 = 0$; no digit c .

Case III: $d = 1$: $10c^2 + c - 156 = 0$; no digit c .

Case IV: $d = 0$: $10c^2 - c - 156 = 0$; here $c = 4$ works.

The only solution is $a = 1$, $b = 9$, $c = 4$, $d = 0$. So Laszlo's mother was born in 1940, Laszlo was born in $19^2 + 40^2 = 1961$, and he is 51 years old in 2012.

6. Consider the sequence $\{a_n\} = \left\{ \frac{2n+12}{n+3} \right\}$ ($n \in \mathbb{N}^+$).

- Find all terms of the sequence that are not less than 3.
- The indices of a_1, a_3, a_9 are three consecutive terms of a geometric sequence. Prove that these three terms of the sequence are three consecutive terms of an arithmetic sequence.
- How many terms of the sequence have the value 2.012 when rounded to three decimal places?
- Find $\lim_{n \rightarrow \infty} a_n$.

(16 points)

Solution: a) The first few terms of the sequence are

$$a_1 = \frac{2 \cdot 1 + 12}{1 + 3} = \frac{7}{2}, \quad a_2 = \frac{2 \cdot 2 + 12}{2 + 3} = \frac{16}{5}, \quad a_3 = \frac{2 \cdot 3 + 12}{3 + 3} = 3$$

These terms qualify. We show that there are no more, since the sequence is strictly decreasing, that is,

$$\frac{2n+12}{n+3} > \frac{2(n+1)+12}{(n+1)+3}, \quad \text{i.e.} \quad \frac{2n+12}{n+3} > \frac{2n+14}{n+4}$$

for every positive integer n . We can multiply both sides by $(n+3)(n+4)$, which is positive for positive integers n :

$$(2n+12)(n+4) > (2n+14)(n+3), \quad 2n^2 + 20n + 48 > 2n^2 + 20n + 42$$

This is indeed true, so the sequence is strictly decreasing. So only the first, second, and third terms of the sequence are not less than 3.

b) The three terms in question are $a_1 = \frac{7}{2}$, $a_3 = 3 = \frac{6}{2}$, $a_9 = \frac{2 \cdot 9 + 12}{9 + 3} = \frac{5}{2}$. Since $\frac{6}{2} - \frac{7}{2} = \frac{5}{2} - \frac{6}{2} = -\frac{1}{2}$, these numbers are indeed three consecutive terms of an arithmetic sequence.

c) We look for the positive integers n for which

$$2.0115 \leq \frac{2n+12}{n+3} < 2.0125$$

We can multiply by $n+3$, which is positive: $2.0115(n+3) \leq 2n+12 < 2.0125(n+3)$, that is, $0.0115n + 6.0345 \leq 12 < 0.0125n + 6.0375$. From the first inequality, $n \leq \frac{5.9655}{0.0115} \approx 518.74$; from the second, $n > 477$. So the possible values of n are 478, 479, ..., 518: 41 terms in all.

d) Using the properties of limits,

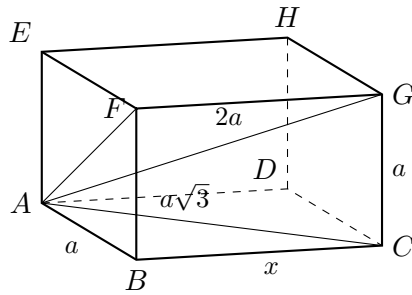
$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n+12}{n+3} = \lim_{n \rightarrow \infty} \frac{2 + \frac{12}{n}}{1 + \frac{3}{n}} = \frac{2+0}{1+0} = 2$$

7. In the cuboid $ABCDEFGH$ the vertices are labeled so that on the face congruent to the base $ABCD$ the vertex E is joined by an edge to A , F to B , G to C , and H to D . We know that $\angle BAF = 45^\circ$ and $\angle CAG = 30^\circ$.

- Prove that $AFGD$ is a square.
- What is the surface area of the tetrahedron $AFBC$, if $AB = a$?
- What is the volume of the tetrahedron $AFHC$, if the distance between AF and HC is $a\sqrt{2}$?

(16 points)

Solution: a) We use the labels of the figure. Since on the side face $ABFE$ of the cuboid the diagonal bisects the angle ($\angle BAF = 45^\circ$), this face is a square. Let its sides be a ; then $AF = a\sqrt{2}$. We know that in the right triangle CAG , $\angle CAG = 30^\circ$ and $CG = a$, so $AG = 2a$ and $AC = a\sqrt{3}$.



By the Pythagorean theorem in the right triangle ABC : $a^2 + x^2 = (a\sqrt{3})^2$, so $x = a\sqrt{2}$. So every edge of the cuboid parallel to this one has length $a\sqrt{2}$. We know that the cross-section $AFGD$ of a cuboid is a rectangle, and now it turned out that two neighboring sides of it are equal: $AF = FG = a\sqrt{2}$. So $AFGD$ is indeed a square.

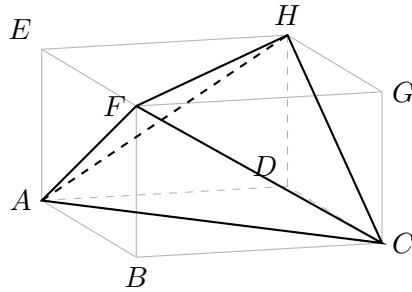
b) The surface of the tetrahedron $AFBC$ consists of three right triangles, ABC , ABF , and CBF , and the isosceles triangle AFC . Here $AC = FC = a\sqrt{3}$, since both are diagonals of congruent rectangles. In the triangle AFC the altitude $TC = m$ (where T is the midpoint of AF) can be found by the Pythagorean theorem:

$$m = \sqrt{3a^2 - \frac{1}{2}a^2} = a\sqrt{\frac{5}{2}}$$

The surface area of the tetrahedron is

$$\begin{aligned} A &= T_{ABC} + T_{ABF} + T_{CBF} + T_{AFC} = a^2 \cdot \frac{\sqrt{2}}{2} + a^2 \cdot \frac{1}{2} + a^2 \cdot \frac{\sqrt{2}}{2} + a^2 \cdot \frac{\sqrt{5}}{2} \\ &= a^2 \cdot \frac{\sqrt{2} + 1 + \sqrt{2} + \sqrt{5}}{2} \approx 3.032a^2 \end{aligned}$$

c) At the vertices B , D , E , and G of the cuboid, three edges of the same lengths meet: a , a , $a\sqrt{2}$. To get the tetrahedron $AFHC$, we must cut off the four tetrahedra spanned by these edges from the cuboid. Since AF and HC lie in opposite faces of the cuboid, their distance is the edge BC ; so if it is $a\sqrt{2}$, then $AB = a$, as in the previous parts.



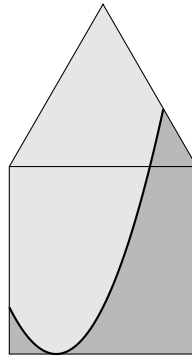
The volume of one tetrahedron cut off is

$$V_{ABCF} = \frac{\frac{a^2}{2} \cdot a\sqrt{2}}{3} = \frac{a^3\sqrt{2}}{6}$$

So the volume of the tetrahedron in the problem is

$$V_{AFHC} = V_{ABCDEFGH} - 4V_{ABCF} = a^3\sqrt{2} - 4 \cdot \frac{a^3\sqrt{2}}{6} = \frac{a^3\sqrt{2}}{3}$$

8. The gable wall of a house consists of a square and an equilateral triangle. The wall is to be plastered in two colors. The boundary line between the two parts will be a parabola, as shown in the figure. The part of the wall above the parabola will be plastered in a lighter color, and the rest in a darker color. What percent of the surface will be the darker shade?

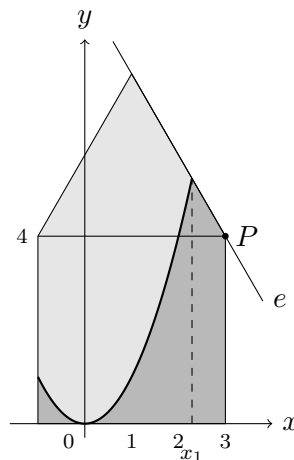


(16 points)

Solution: Let the side of the square be 4. Then the area of the gable wall is

$$T = 4^2 + \frac{4^2\sqrt{3}}{4} = 16 + 4\sqrt{3} \approx 22.93$$

We find the area of the darker part. For this we place the figure in a coordinate system, as in the figure: the parabola is the graph of the function $f(x) = x^2$, and the wall lies between $x = -1$ and $x = 3$.



The line e passes through the point $P(3, 4)$, and since its angle of inclination is -60° , its slope is $-\sqrt{3}$. So the equation of the line e is $y - 4 = -\sqrt{3}(x - 3)$, that is, e is the graph of the function $g(x) = -\sqrt{3}x + 3\sqrt{3} + 4$. We find the intersection of the parabola and the line e ; the solution of the equation $f(x) = g(x)$ gives its first coordinate:

$$x^2 = -\sqrt{3}x + 3\sqrt{3} + 4, \quad x^2 + \sqrt{3}x - (3\sqrt{3} + 4) = 0$$

One of the two roots is negative; we need the intersection with positive first coordinate: $x_1 \approx 2.288$. The area in question is

$$\begin{aligned} T_1 &= \int_{-1}^{x_1} x^2 dx + \int_{x_1}^3 (-\sqrt{3}x + 3\sqrt{3} + 4) dx = \left[\frac{x^3}{3} \right]_{-1}^{x_1} + \left[-\sqrt{3} \cdot \frac{x^2}{2} + (3\sqrt{3} + 4)x \right]_{x_1}^3 \\ &\approx 7.61 \end{aligned}$$

This is about $\frac{7.61}{22.93} \approx 33.2\%$ of the whole wall.

Remark: Instead of the second integral we could use the formula for the area of a trapezoid.

9. Find the real numbers x for which the sum of the squares of $\cos x$ and $\cos 2x$ equals the square of $\cos 3x$.

(16 points)

Solution: We must solve the equation $\cos^2 x + \cos^2 2x = \cos^2 3x$. It can be written as

$$\cos^2 2x = (\cos 3x - \cos x)(\cos 3x + \cos x)$$

Transforming further with the sum-to-product formulas:

$$\cos^2 2x = \left(-2 \sin \frac{3x+x}{2} \sin \frac{3x-x}{2} \right) \left(2 \cos \frac{3x+x}{2} \cos \frac{3x-x}{2} \right)$$

$$\cos^2 2x = -2 \sin 2x \sin x \cdot 2 \cos 2x \cos x$$

$$\cos^2 2x = -2 \sin 2x \cdot \sin 2x \cdot \cos 2x$$

$$\cos 2x (\cos 2x + 2 \sin^2 2x) = 0$$

$$\cos 2x [\cos 2x + 2(1 - \cos^2 2x)] = 0$$

$$\cos 2x (2 \cos^2 2x - \cos 2x - 2) = 0$$

We get three possible values for $\cos 2x$:

Case I: $\cos 2x = 0$, so $x = \frac{\pi}{4} + k \cdot \frac{\pi}{2}$, $k \in \mathbb{Z}$.

Case II: $\cos 2x = \frac{1+\sqrt{17}}{4} > 1$, so this gives no solution.

Case III: $\cos 2x = \frac{1-\sqrt{17}}{4} \approx -0.7808$, so $2x \approx \pm 2.4667 + 2k\pi$, that is, $x \approx 1.2334 + k\pi$ or $x \approx 1.9082 + k\pi$, $k \in \mathbb{Z}$.

So all real numbers satisfying the conditions are given by these three families.