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translated from: KoMaL, April 2013, pages 208-209

Pdf file: KoMaL, April 2013, pages 208-209

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. In the school chess championship everybody played exactly once against everybody else. When 42 games had been played, everybody still had four games left. How many players took part in this championship?

(11 points)

Solution: Let the number of players be n . The total number of games is $\frac{n(n-1)}{2}$. If everybody still has 4 games left, that means $\frac{4n}{2}$ games still to be played altogether. So the number of games played is

$$\frac{n(n-1)}{2} - \frac{4n}{2} = 42$$

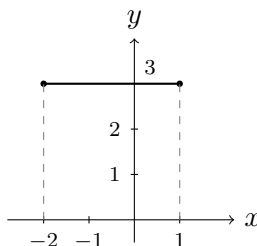
which gives the quadratic equation $n^2 - 5n - 84 = 0$, with roots $n_1 = 12$, $n_2 = -7$. So 12 players took part in the championship.

2. Graph the following functions.

- a) $f(x) = |x+2| + |x-1|$ on the interval $[-2, 1]$;
 b) $g(x) = (\tan x + \cot x) \sin x \cos x$ on the interval $[-\pi, \pi]$;
 c) $h(x) = \log_{2013} x \cdot \log_x 2014 \cdot \log_{2014} 2013$ on the interval $]0, 3]$.

(13 points)

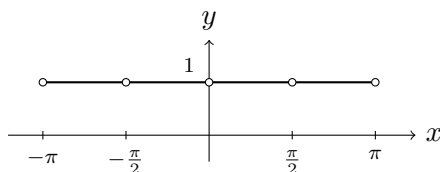
Solution: a) On the given interval $[-2, 1]$ the rule of f can be written as $f(x) = (x+2) - (x-1) = 3$. So on this interval the function is constant.



- b) The rule of g can be transformed:

$$g(x) = \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \right) \sin x \cos x = \sin^2 x + \cos^2 x = 1$$

Because of $\tan x$ and $\cot x$, $x \neq \frac{k\pi}{2}$, where k is an integer. In the interval $[-\pi, \pi]$ these are the values for $k = -2, -1, 0, 1, 2$; so g is not defined at $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2},$ and π , and otherwise it is constant 1.

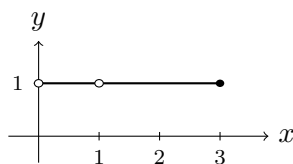


Note: KoMaL's text says that g is not defined at $-\pi$, 0 , and π ; it is also undefined at $\pm\frac{\pi}{2}$ (where $\tan x$ is undefined), as its figure shows.

c) By the definition of the logarithm, x is a positive real number not equal to 1. Writing every logarithm in base 2013:

$$h(x) = \log_{2013} x \cdot \frac{\log_{2013} 2014}{\log_{2013} x} \cdot \frac{\log_{2013} 2013}{\log_{2013} 2014} = \log_{2013} 2013 = 1$$

So h is constant 1 on $]0, 1[$ and on $]1, 3]$.



3. Solve the following equation, where n is an arbitrary positive integer greater than 1.

$$\sqrt{x^2 + x - 2} + \sqrt{x^2 + 2x - 3} + \cdots + \sqrt{x^2 + nx - (n+1)} = 0$$

(13 points)

Solution: The left side is a sum of n terms, each a nonnegative real number. A sum of nonnegative numbers can only be zero if each of them is zero.

We find the zeros of the quadratic expressions under the square roots. Consider the general term $x^2 + kx - (k+1)$. The sum of its two zeros is $-k$ and their product is $-(k+1)$, so the zeros are $x_1 = -(k+1)$ and $x_2 = 1$. So for every k , one zero of the quadratic expression is 1, and $x = 1$ makes all the terms zero (while every other zero of a term makes some other term positive).

So the only solution of the equation is $x = 1$.

4. Write the equation of the circle that passes through the point $A(-7, 5)$ and touches the circle $(x-2)^2 + (y-3)^2 = 17$ at the point $E(-2, 2)$.

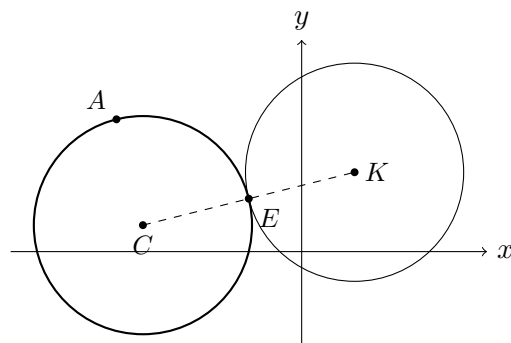
(14 points)

Solution: The center of the given circle is $K(2, 3)$. The center of the circle in question lies on the line through $E(-2, 2)$ and $K(2, 3)$. A direction vector of this line is $\overrightarrow{EK} = (4, 1)$, so its equation is $x - 4y = -10$.

AE is a chord of the circle in question, so its center also lies on the perpendicular bisector of the chord determined by $A(-7, 5)$ and $E(-2, 2)$. The midpoint of the chord is $F(-\frac{9}{2}, \frac{7}{2})$, and a normal vector of the perpendicular bisector is $\overrightarrow{AE} = (5, -3)$, so its equation is $5x - 3y = -33$. Solving the system $x - 4y = -10$, $5x - 3y = -33$, the center of the circle is $C(-6, 1)$. Its radius is the distance of $A(-7, 5)$ and $C(-6, 1)$:

$$r = AC = \sqrt{(-7+6)^2 + (5-1)^2} = \sqrt{17}$$

So the equation of the circle is $(x+6)^2 + (y-1)^2 = 17$.



Part II

5. The base AB of the isosceles triangle ABC is 26 cm, and its legs are 85 cm long. Let F be the midpoint of the base AB , K the center of the inscribed circle, O the center of the circumscribed circle, S the centroid, and M the orthocenter.
- What is the perimeter of the triangle ASO ?
 - How long is the segment FK ?
 - How long is the segment MF , and at what angle is the leg AC seen from the point M ?

(16 points)

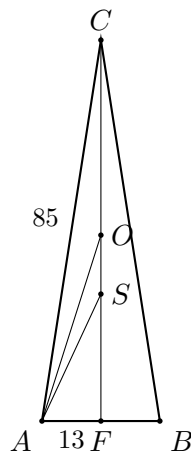
Solution: a) We get the length of the side OS of the triangle ASO as the difference of the segments FO and FS . The points F , K , O , S , M , and C of the problem lie on one line, the altitude from the vertex C , since the triangle ABC is isosceles. We find the length of this altitude by the Pythagorean theorem in the right triangle ACF : $13^2 + CF^2 = 85^2$, so $CF = 84$ cm. Since the centroid is the point of the median dividing it into three parts closer to the side (and now CF is a median), $FS = 28$ cm.

The circumcenter is at distance $84 - r$ from the line of the base, where r is the circumradius. By the Pythagorean theorem in the triangle AFO ,

$$13^2 + (84 - r)^2 = r^2, \quad 13^2 + 84^2 - 2 \cdot 84r = 0, \quad \text{so} \quad r = \frac{7225}{168} \approx 43.01 \text{ cm}$$

So $FO = 84 - \frac{7225}{168} = \frac{6887}{168} \approx 40.99$ cm, and $OS = FO - FS = \frac{2183}{168} \approx 12.99$ cm. $AO = r$, and the length of AS comes from the right triangle ASF by the Pythagorean theorem: $AS^2 = 13^2 + 28^2$, so $AS = \sqrt{953} \approx 30.87$ cm. So the perimeter of the triangle ASO is

$$k = OS + AS + AO \approx 12.99 + 30.87 + 43.01 = 86.87 \text{ cm}$$



b) The incenter is at distance ρ from the base (where ρ is the inradius), so we find ρ by writing the area of the triangle ABC in two ways: $t = \rho s = \frac{c \cdot m_c}{2}$, where $s = \frac{2 \cdot 85 + 26}{2} = 98$ is half the perimeter. Then $98\rho = 1092$, so

$$FK = \rho = \frac{1092}{98} = \frac{78}{7} \approx 11.14 \text{ cm}$$

c) Let AT be the altitude to the leg BC . Writing twice the area of the triangle ABC in two ways: $26 \cdot 84 = 85 \cdot AT$, so $AT = \frac{2184}{85}$. By the Pythagorean theorem in the right triangle ATB :

$$TB = \sqrt{26^2 - \left(\frac{2184}{85}\right)^2} = \frac{\sqrt{114\,244}}{85} = \frac{338}{85}$$

Since the triangles AFM and ATB are similar (their angles are equal in pairs), the ratios of the corresponding sides are equal:

$$\frac{MF}{13} = \frac{TB}{AT} = \frac{338}{2184}, \quad \text{so} \quad MF = \frac{169}{84} \approx 2.01 \text{ cm}$$

The angle CMA in question is the exterior angle at M of the right triangle AMF , so we find the interior angle δ at M : $\tan \delta = \frac{13}{\frac{169}{84}} = \frac{84}{13}$, so $\delta \approx 81.2^\circ$. So the angle CMA is about 98.8° .

6. Solve the following system of equations over the set of pairs of integers.

$$\begin{aligned} 3^{3x-1} &= 2y^3 - 11y - 693 \\ x &= \log_3(y+1) \end{aligned}$$

(16 points)

Solution: By the second equation, $y+1 > 0$ and $3^x = y+1$, that is, $y = 3^x - 1$. Then

$$3^{3x-1} = \frac{(3^x)^3}{3} = \frac{(y+1)^3}{3}$$

Substituting into the first equation:

$$\frac{(y+1)^3}{3} = 2y^3 - 11y - 693, \quad 0 = 5y^3 - 3y^2 - 36y - 2080$$

Since x and y are integers and $y+1 = 3^x > 0$, y is a nonnegative integer that is 1 less than a power of 3. Write the cubic equation in the form $2080 = y(5y^2 - 3y - 36)$. The factor in parentheses is an integer, so y must be a divisor of 2080. So the possible values of y are 2, 8, 26, 80. Substituting these into the cubic equation, only $y = 8$ is a root. Then $x = 2$. So the only solution of the system is $x = 2$, $y = 8$.

Remark: Knowing $y = 8$, the cubic equation can be written as $0 = (y-8)(5y^2 + 37y + 260)$. This also shows that the system has only one solution, since the discriminant of the quadratic factor is negative.

7. a) Find the integral of the function $f(x) = x^2 - 10x + 27$ on the interval $[3, 6]$.
 b) By how much must the given interval be shifted so that the integral is 18?

(16 points)

Solution: a)

$$\int_3^6 (x^2 - 10x + 27) dx = \left[\frac{x^3}{3} - 5x^2 + 27x \right]_3^6 = (72 - 180 + 162) - (9 - 45 + 81) = 9$$

- b) Let the shift be d in the negative direction. Then

$$\begin{aligned} \int_{3-d}^{6-d} (x^2 - 10x + 27) dx &= \left[\frac{x^3}{3} - 5x^2 + 27x \right]_{3-d}^{6-d} \\ &= \frac{(6-d)^3}{3} - 5(6-d)^2 + 27(6-d) - \left(\frac{(3-d)^3}{3} + 5(3-d)^2 - 27(3-d) \right) \\ &= 3d^2 + 3d + 9 = 18 \end{aligned}$$

So we must solve the quadratic equation $d^2 + d - 3 = 0$:

$$d_{1,2} = \frac{-1 \pm \sqrt{13}}{2}$$

So the interval can be shifted to the left by $d_1 = \frac{-1+\sqrt{13}}{2} \approx 1.3$ units, or to the right by $|d_2| = \frac{1+\sqrt{13}}{2} \approx 2.3$ units.

Remark: From the answer to part a) we see that in part b) the task is to double the integral by the shift. The value 18 also appears so that those who answered the first question incorrectly can still answer this question correctly.

8. The following data set is given: 1, 1, 2, 4, 8, 8, 8, 9, 13.
- a) Prove that for this data set, the sum of the distances from the mean of the numbers smaller than the mean equals the sum of the distances from the mean of the numbers greater than it.
- b) Find the average of the data set for which the sum of the absolute distances of the data from it is minimal.
- c) Find the average of the data set for which the sum of the squares of the distances of the data from it is minimal.

(16 points)

Solution: a) The mean of the data set is

$$\bar{x} = \frac{1 + 1 + 2 + 4 + 8 + 8 + 8 + 9 + 13}{9} = 6$$

The sum of the distances from it of the numbers smaller than the mean is $(6-1)+(6-1)+(6-2)+(6-4) = 16$, and the sum of the distances of the numbers greater than the mean is $(8-6)+(8-6)+(8-6)+(9-6)+(13-6) = 16$. So the two sums are indeed equal.

- b) We look for the number x for which the following sum is minimal:

$$f(x) = |1-x| + |1-x| + |2-x| + |4-x| + |8-x| + |8-x| + |8-x| + |9-x| + |13-x|$$

Group the terms in pairs, from the outside in:

$$\begin{aligned} f_1(x) &= |1-x| + |13-x|, & f_2(x) &= |1-x| + |9-x|, & f_3(x) &= |2-x| + |8-x| \\ f_4(x) &= |4-x| + |8-x|, & f_5(x) &= |8-x| \end{aligned}$$

For example, $f_1(x) = -2x + 14$ if $x < 1$, $f_1(x) = 12$ if $1 \leq x \leq 13$, and $f_1(x) = 2x - 14$ if $13 < x$; so the set of minimum points of f_1 is $[1, 13]$. Similarly, the sets of minimum points of f_2, f_3, f_4, f_5 are $[1, 9]$, $[2, 8]$, $[4, 8]$, and $\{8\}$. Since

$$f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x) \quad \text{and} \quad [1, 13] \supset [1, 9] \supset [2, 8] \supset [4, 8] \supset \{8\}$$

all five functions have their minimum at 8, so the minimum point in question is 8. We also found that this number is exactly the median of the data set.

c) We look for the number x for which the following sum is minimal:

$$f(x) = (1-x)^2 + (1-x)^2 + (2-x)^2 + \cdots + (13-x)^2$$

It can be written as

$$f(x) = 9x^2 - 2(1 + 1 + 2 + 4 + 8 + 8 + 8 + 9 + 13)x + (1^2 + 1^2 + 2^2 + 4^2 + 8^2 + 8^2 + 8^2 + 9^2 + 13^2)$$

The minimum point of a quadratic function $ax^2 + bx + c$ (with $a > 0$) is $x = -\frac{b}{2a}$. Here

$$x = \frac{2(1 + 1 + 2 + 4 + 8 + 8 + 8 + 9 + 13)}{2 \cdot 9} = \frac{54}{9} = 6$$

So the minimum point in question is 6; it is exactly the mean of the data set.

9. Find the acute angle x for which the expression

$$\frac{4 \sin^2 x - 4 \sin^2 x \cos^2 x + 1}{\sin^2 x}$$

is minimal. What is the smallest value?

(16 points)

Solution: For acute angles the expression is defined. We transform it:

$$\frac{4 \sin^2 x - 4 \sin^2 x \cos^2 x + 1}{\sin^2 x} = 4 - 4 \cos^2 x + \frac{1}{\sin^2 x} = 4 \sin^2 x + \frac{1}{\sin^2 x}$$

For every acute angle, $4 \sin^2 x$ and $\frac{1}{\sin^2 x}$ are positive real numbers. By the inequality between the geometric and arithmetic means of two positive numbers, $\sqrt{ab} \leq \frac{a+b}{2}$, with equality exactly when $a = b$. Applying it to $a = 4 \sin^2 x$ and $b = \frac{1}{\sin^2 x}$:

$$4 = 2\sqrt{4 \sin^2 x \cdot \frac{1}{\sin^2 x}} \leq 4 \sin^2 x + \frac{1}{\sin^2 x}$$

So the minimum value is 4, which occurs when $4 \sin^2 x = \frac{1}{\sin^2 x}$. Then $4 \sin^4 x = 1$, so $\sin^2 x = \frac{1}{2}$ and $\sin x = \frac{\sqrt{2}}{2}$ (since x is acute), that is, $x = \frac{\pi}{4}$.

Note: KoMaL writes “ $4 \sin^4 x = 1$, that is, $\sin x = \pm 1$ ”; it should be $\sin^2 x = \frac{1}{2}$. Its final answer $x = \frac{\pi}{4}$ is correct.

Remark: $2\left(2 \sin^2 x + \frac{1}{2 \sin^2 x}\right)$ is twice the sum of a positive number and its reciprocal, which is minimal when the number is 1.