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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. In a television talent show, the singers of four mentors compete. The ranking is decided by the phone votes of the viewers. To make it more exciting, the host gives each mentor a little information about the state of the vote. These statements were made, in this order:

   Of your three contestants, exactly two are not in one of the last three places.

   Of your two contestants, exactly one is in one of the last three places.

   Of your two contestants, at least one is in one of the last three places.

   Of your two contestants, at least one is certainly not in one of the last three places.

   a) Can any of the four mentors think now that none of his or her contestants is in one of the last three places?

   b) How many rankings are possible, based on the statements made?

   (11 points)

   Solution: a) By the first statement, exactly one contestant of the first mentor is in one of the last three places. Similarly, exactly one contestant of the second mentor is in one of the last three places. One or two contestants of the third mentor could be in these places, but by the first two statements only one place is left, so exactly one of them is there. Then the fourth mentor's statement "at least one is certainly not in one of the last three places" can only happen if neither of his or her contestants is in one of the last three places. So the fourth mentor can relax: none of his or her contestants is in one of the last three places.

   b) Let the three contestants of the first mentor be A_1, A_2, A_3, those of the second mentor B_1, B_2, those of the third mentor C_1, C_2, and those of the fourth mentor D_1, D_2. We need to count all the rankings of these nine contestants that agree with the statements.

   We know that one contestant each of the first, second, and third mentors is in the last three places (for example A_1, B_1, C_1). The other six contestants can be ahead of them in 6! orders, and the three contestants in the last three places can be in 3! orders. That is $6! \cdot 3! = 4320$ cases. We also have several choices for the three contestants who did poorly: they can be chosen in $3 \cdot 2 \cdot 2 = 12$ ways. So the number of all possible rankings is $12 \cdot 4320 = 51\,840$.

2. Solve the following equation. (lg denotes the logarithm in base 10.)

$$\lg(2-x) \cdot \sqrt{6-x-x^2} \cdot \sin\left(x - \frac{\pi}{6}\right) = 0$$

   (12 points)

   Solution: Because of the logarithm, $2 - x > 0$, so $x < 2$. Because of the square root, $6 - x - x^2 \geq 0$, that is, $(2-x)(x+3) \geq 0$, so $-3 \leq x \leq 2$. So the domain of the equation is $[-3, 2)$.

A product is zero exactly when one of its factors is zero. We look at the zeros of the three factors one by one.

The zero of the first factor is $x = 1$, and it is in the domain.

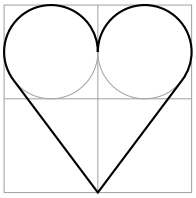
The zeros of the second factor are -3 and 2. Of these, only -3 is in the domain.

The zeros of the third factor are $\frac{\pi}{6} + k\pi$, where k is an integer. Because of the domain, only $k = -1$ and $k = 0$ give suitable values.

So the solutions of the equation are

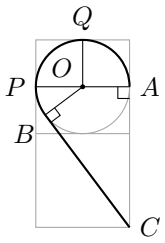
$$-3, \quad -\frac{5\pi}{6}, \quad \frac{\pi}{6}, \quad 1$$

3. There is a heart shape on Enikő's T-shirt. The heart can be drawn into a square with 24 cm sides, as shown in the figure. What are the perimeter and the area of this heart?



(14 points)

Solution: The figure is symmetric about a vertical line, so it is enough to work with its left half.



The radius of the circles is a quarter of the side of the big square, that is, 6 cm. The two tangent segments drawn to a circle from an outside point have the same length, so $CA = CB$. The segment CA is three quarters of the side of the big square. So $CA = CB = 18$ cm. The quadrilateral $CAOB$ is a kite with two right angles (at A and at B); it consists of two right triangles with legs 6 cm and 18 cm, so its area is

$$T_1 = 6 \cdot 18 = 108 \text{ cm}^2$$

In the right triangle OBC, for the acute angle α at O we have $\tan\alpha = \frac{18}{6} = 3$, so $2\alpha \approx 143.13°$.

The central angle of the arc AB (through P and Q) is $360° - 143.13° = 216.87°$. The area of the corresponding sector is

$$T_2 = 6^2\pi \cdot \frac{216.87°}{360°} \approx 68.13 \text{ cm}^2$$

The area of the heart is

$$2\,(T_1 + T_2) \approx 2\,(108 + 68.13) = 352.26 \text{ cm}^2$$

The length of the arc AB is

$$i = 2 \cdot 6\pi \cdot \frac{216.87°}{360°} \approx 22.71 \text{ cm}$$

The perimeter of the heart is

$$K = 2\,(i + BC) \approx 2\,(22.71 + 18) = 81.42 \text{ cm}$$

4. We chose eight cities such that there is a direct flight between any two of them. On a winter day, because of extreme weather, flights are suspended on nine of all these routes. What is the probability that of four randomly chosen routes, there are flights on exactly two?

(14 points)

Solution: Since there is a direct flight between any two of the eight cities, there are $\binom{8}{2} = 28$ routes altogether. First we count all cases. We choose four of the 28 routes at random; this can be done in $\binom{28}{4}$ ways. Next we count the favorable cases. The two routes without flights can be chosen from the nine suspended ones, and the two routes with flights can be chosen from the other 19. So the number of favorable cases is $\binom{9}{2}\binom{19}{2}$.

The probability is the number of favorable cases divided by the number of all cases:

$$p = \frac{\binom{9}{2}\binom{19}{2}}{\binom{28}{4}} = \frac{36 \cdot 171}{20\,475} = \frac{6156}{20\,475} \approx 0.30$$

Part II

5. Solve the following equation over the set of real numbers.

$$\left(\frac{\sqrt{5}}{4}\right)^{\frac{12-2\lg x}{-5+\lg x} - \frac{4}{1+\lg x}} = 10.24$$

(16 points)

Solution: Only positive numbers have a logarithm, so $x > 0$. Also, the denominators cannot be zero, so $x \neq 10^5$ and $x \neq 10^{-1}$. So the domain of the equation is the set of positive numbers except 10^{-1} and 10^5.

Since

$$10.24 = \frac{1024}{100} = \frac{256}{25} = \frac{4^4}{\sqrt{5}^4} = \left(\frac{\sqrt{5}}{4}\right)^{-4}$$

the equation can be written as

$$\left(\frac{\sqrt{5}}{4}\right)^{\frac{12-2\lg x}{-5+\lg x} - \frac{4}{1+\lg x}} = \left(\frac{\sqrt{5}}{4}\right)^{-4}$$

The exponential function is one-to-one, so the exponents are equal:

$$\begin{aligned} \frac{12-2\lg x}{-5+\lg x} - \frac{4}{1+\lg x} &= -4 \\ (12-2\lg x)(1+\lg x) - 4(-5+\lg x) &= -4(-5+\lg x)(1+\lg x) \\ \lg^2 x - 5\lg x + 6 &= 0 \end{aligned}$$

This is a quadratic equation in $\lg x$; its two solutions are $\lg x_1 = 2$ and $\lg x_2 = 3$. So $x_1 = 100$ and $x_2 = 1000$. Both are in the domain, so they are indeed solutions of the equation.

6. We cut the cube $ABCDEFGH$ into two pieces with the plane through the vertex A, the midpoint P of the edge BF, and the center Q of the face $ADHE$.

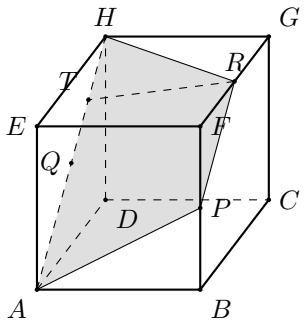
a) Find the area of the cross-section in terms of the edge of the cube.

b) What is the ratio of the volumes of the two pieces?

(16 points)

Solution: a) Let the edge of the cube be a. The line AQ lies in the cutting plane, and it passes through H (the diagonals of the square $ADHE$ bisect each other). So H is also on the cross-section. The faces $ADHE$ and $BCGF$ of the cube

are parallel, so the cutting plane meets them in parallel lines. One of these lines is AH; the line through P parallel to it meets the edge FG at its midpoint R. So the cross-section is the quadrilateral $APRH$, an isosceles trapezoid. We need to find its area.



$AH = a\sqrt{2}$, because it is a diagonal of a square with side a. $PR = \frac{a\sqrt{2}}{2}$, because it is half of the diagonal of such a square. We find the legs $HR = AP$ in the right triangle HGR by the Pythagorean theorem:

$$HR = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} = \frac{a\sqrt{5}}{2}$$

Let T be the foot of the perpendicular from R to AH. In the right triangle RTH,

$$TH = \frac{AH - PR}{2} = \frac{a\sqrt{2}}{4}$$

By the Pythagorean theorem, the height TR of the trapezoid $APRH$ is

$$TR = \sqrt{HR^2 - TH^2} = \sqrt{\frac{5}{4}a^2 - \frac{1}{8}a^2} = a \cdot \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}a$$

The area of the cross-section is

$$T = \frac{\left(a\sqrt{2} + \frac{a\sqrt{2}}{2}\right) \cdot \frac{3\sqrt{2}}{4}a}{2} = \frac{9}{8}a^2$$

b) One of the pieces is a frustum of a pyramid whose bases are the triangles AEH and PFR. (The lines AP, EF, and HR meet in one point, because $PF = \frac{AE}{2}$, $FR = \frac{EH}{2}$, and the two triangles are in parallel planes.) The area of the base AEH, an isosceles right triangle, is $T = \frac{a \cdot a}{2} = \frac{a^2}{2}$, and the area of the top PFR is

$$t = \frac{\frac{a}{2} \cdot \frac{a}{2}}{2} = \frac{a^2}{8}$$

The distance between the two planes is a; this is the height m of the frustum. By the formula for the volume of a frustum,

$$V_1 = \frac{m}{3}\left(T + \sqrt{Tt} + t\right) = \frac{a}{3}\left(\frac{a^2}{2} + \sqrt{\frac{a^2}{2} \cdot \frac{a^2}{8}} + \frac{a^2}{8}\right) = \frac{a}{3} \cdot \frac{7a^2}{8} = \frac{7}{24}a^3$$

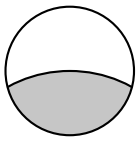
The volume of the other piece is

$$V_2 = a^3 - \frac{7}{24}a^3 = \frac{17}{24}a^3$$

So the ratio of the volumes of the two pieces is

$$\frac{V_1}{V_2} = \frac{\frac{7}{24}a^3}{\frac{17}{24}a^3} = \frac{7}{17}$$

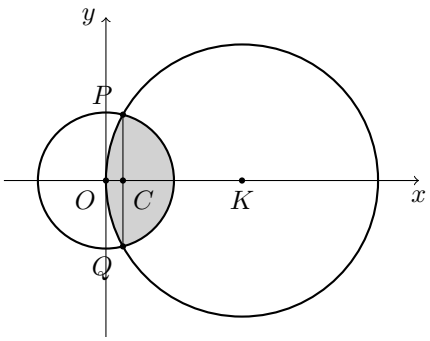
7. The figure shows a sketch of one of Ibolya's earrings. The circular arc that passes through the center of the circular disk has twice the radius of the disk. What percent of the earring is darker?



(16 points)

Solution: We may assume that the radius of the earring is 1 and the radius of the arc is 2. We place the figure in a coordinate system so that the center O of the disk is the origin, and the center of the arc is the point $K\,(2,0)$. (So the figure is turned compared to the figure of the problem.) The equations of the two circles are

$$x^2+y^2=1, \qquad (x-2)^2+y^2=4$$



To find the intersection points of the two circles, we solve this system of equations. We expand the square in the second equation and replace x^2+y^2 by 1:

$$-4x+4+1=4, \qquad \text{so} \qquad x=\frac{1}{4}$$

Substituting this into the first equation, we get the two intersection points:

$$P\left(\frac{1}{4},\frac{\sqrt{15}}{4}\right), \qquad Q\left(\frac{1}{4},-\frac{\sqrt{15}}{4}\right)$$

The segment PQ cuts the darker part of the earring into two circular segments. We find their areas separately. We use the formula for the area of a circular segment, $T=\frac{1}{2}\left[ri-h\,(r-m)\right]$, where r is the radius of the arc, i is the length of the arc, h is the length of the chord, and m is the height of the segment. (This is the area of the sector minus the area of the triangle formed by the chord and the center.)

For the segment of the circle of radius 1 we already know

$$r_1=1, \qquad h_1=PQ=2\cdot\frac{\sqrt{15}}{4}=\frac{\sqrt{15}}{2}, \qquad m_1=\frac{3}{4}$$

We still need the length of the arc PQ, so we need the angle POQ. Let C be the intersection point of PQ and the x-axis. In the right triangle POC, for the angle β at O,

$$\tan\beta=\frac{\frac{\sqrt{15}}{4}}{\frac{1}{4}}=\sqrt{15}$$

so $\angle POQ=2\beta\approx 151.04^\circ$. So

$$\frac{i_1}{2\pi}=\frac{151.04^\circ}{360^\circ}, \qquad \text{that is,} \qquad i_1\approx 2.636$$

Substituting into the area formula:

$$T_1 = \frac{1}{2}\left[1 \cdot 2.636 - \frac{\sqrt{15}}{2}\left(1 - \frac{3}{4}\right)\right] \approx 1.076$$

For the segment of the circle of radius 2 we have

$$r_2 = 2, \quad h_2 = h_1 = \frac{\sqrt{15}}{2}, \quad m_2 = \frac{1}{4}$$

Again we need the length of the arc PQ, now on the circle of radius 2, so we need the angle PKQ. In the right triangle PKC, for the angle α at K,

$$\tan\alpha = \frac{\frac{\sqrt{15}}{4}}{\frac{7}{4}} = \frac{\sqrt{15}}{7}$$

so $\angle PKQ = 2\alpha \approx 57.91°$. So

$$\frac{i_2}{4\pi} = \frac{57.91°}{360°}, \quad \text{that is,} \quad i_2 \approx 2.021$$

Substituting into the area formula again:

$$T_2 = \frac{1}{2}\left[2 \cdot 2.021 - \frac{\sqrt{15}}{2}\left(2 - \frac{1}{4}\right)\right] \approx 0.327$$

The area of the darker part is $T_1 + T_2 \approx 1.076 + 0.327 = 1.403$, and the area of the earring is π. Since $\frac{1.403}{\pi} \approx 0.447$, about 44.7% of the earring is darker.

8. About a cubic function f we know that $f(2) = f(4) = 4$, the coefficient of its cubic term is 1, and

$$\int_2^6 f(x)\, dx = 16$$

Find the equation of the tangent line to the graph of f at the point with x-coordinate 5.

(16 points)

Solution: The cubic function is $f(x) = x^3 + bx^2 + cx + d$. We know that

$$\begin{aligned} f(2) &= 8 + 4b + 2c + d = 4 \\ f(4) &= 64 + 16b + 4c + d = 4 \end{aligned}$$

Also,

$$\begin{aligned} \int_2^6 f(x)\, dx &= \int_2^6 \left(x^3 + bx^2 + cx + d\right) dx = \left[\frac{x^4}{4} + b \cdot \frac{x^3}{3} + c \cdot \frac{x^2}{2} + d \cdot x\right]_2^6 = \\ &= \left(\frac{6^4}{4} + b \cdot \frac{6^3}{3} + c \cdot \frac{6^2}{2} + d \cdot 6\right) - \left(\frac{2^4}{4} + b \cdot \frac{2^3}{3} + c \cdot \frac{2^2}{2} + d \cdot 2\right) = \\ &= (324 + 72b + 18c + 6d) - \left(4 + \frac{8}{3}b + 2c + 2d\right) = \\ &= \frac{208}{3}b + 16c + 4d + 320 = 16 \end{aligned}$$

Rearranging the three equations, we get the system

$$\begin{aligned} 4b + 2c + d &= -4 \\ 16b + 4c + d &= -60 \\ 208b + 48c + 12d &= -912 \end{aligned}$$

Its solution is $b = -12$, $c = 44$, $d = -44$, so the cubic function is

$$f(x) = x^3 - 12x^2 + 44x - 44$$

We find the slope of the tangent line. Since $f'(x) = 3x^2 - 24x + 44$, we have $f'(5) = 3 \cdot 5^2 - 24 \cdot 5 + 44 = -1$. The slope of the tangent line is $m = -1$.

Since $f(5) = 5^3 - 12 \cdot 5^2 + 44 \cdot 5 - 44 = 1$, the point of tangency is $E(5, 1)$.

The equation of the tangent line through $E(5, 1)$ with slope $m = -1$ is $y = -x + 6$, that is, $x + y = 6$.

9. Solve the following equation.

$$\left(x^2 - 2013^2\right)^2 - 1 = 8052x$$

(16 points)

Solution: We expand the square and add 1 to both sides:

$$x^4 - 2 \cdot 2013^2 \cdot x^2 + 2013^4 = 8052x + 1$$

We add $4 \cdot 2013^2 \cdot x^2$ to both sides. Then the left side is still a perfect square, and the right side also becomes a perfect square (note that $8052 = 4 \cdot 2013$):

$$\begin{aligned} \left(x^2 + 2013^2\right)^2 &= 4 \cdot 2013^2 \cdot x^2 + 8052x + 1 \\ \left(x^2 + 2013^2\right)^2 &= (2 \cdot 2013x + 1)^2 \end{aligned}$$

We bring everything to the left side and factor with the identity $a^2 - b^2 = (a - b)(a + b)$:

$$\begin{aligned} \left(x^2 + 2013^2\right)^2 - (2 \cdot 2013x + 1)^2 &= 0 \\ \left(x^2 + 2013^2 - 2 \cdot 2013x - 1\right)\left(x^2 + 2013^2 + 2 \cdot 2013x + 1\right) &= 0 \end{aligned}$$

Both factors can be rewritten:

$$\Big[(x - 2013)^2 - 1\Big] \cdot \Big[(x + 2013)^2 + 1\Big] = 0$$

The first factor can be factored further:

$$\begin{aligned} (x - 2013 - 1) \cdot (x - 2013 + 1) \cdot \Big[(x + 2013)^2 + 1\Big] &= 0 \\ (x - 2014) \cdot (x - 2012) \cdot \Big[(x + 2013)^2 + 1\Big] &= 0 \end{aligned}$$

The third factor is positive for every x. The first two factors give the solutions $x_1 = 2014$ and $x_2 = 2012$. All our steps were equivalent transformations, so these are the solutions of the original equation.

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