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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. How many different cuboids are there whose edge lengths are integers, whose volume is 15 000 units, and for which the greatest common divisor of the lengths of the three edges is 5?

(12 points)

Solution: The greatest common divisor of the edge lengths is 5, so the three edges are $5a$, $5b$, and $5c$, where $\gcd(a, b, c) = 1$. Since $\frac{15\,000}{5^3} = 120$, we have $abc = 120$. So the question is how many triples of positive integers a, b, c have greatest common divisor 1 and product 120; since the order of the numbers does not matter, we may assume $a \leq b \leq c$. We know that $120 = 2^3 \cdot 3 \cdot 5$.

If $a = 1$, then $bc = 120$; since 120 has $d(120) = 4 \cdot 2 \cdot 2 = 16$ divisors and 120 is not a perfect square, there are $\frac{16}{2} = 8$ suitable triples.

If $a = 2$, then $bc = 60$, and we only need the cases $b, c \geq 2$, $b \leq c$. If both b and c were divisible by 2, then $\gcd(a, b, c) = 2$. So only one of them is divisible by 2, and then by 4. One possibility is $b = 4b'$, $b'c = 15$; this is only possible with $b = 4$, $c = 15$. The other possibility is $c = 4c'$, and then $b = 3$, $c = 20$ or $b = 5$, $c = 12$. So there are 3 suitable triples.

If $a = 3$, then $bc = 40 = 2^3 \cdot 5$, $b, c \geq 4$, $b \leq c$: $b = 4$, $c = 10$ or $b = 5$, $c = 8$.

If $a = 4$, the only solution is $b = 5$, $c = 6$.

There are no more possibilities, since $5^3 > 120$. So altogether there are $8 + 3 + 2 + 1 = 14$ suitable cuboids.

2. Solve the following equations over the real numbers.

a) $\sqrt{x - \sqrt{x + 2}} = 2$

b) $\log_2 x + \log_4 x + \log_8 x = \log_{16} x^3$

(13 points)

Solution: a) The domain of the equation follows from the inequalities $x \geq \sqrt{x + 2}$ and $x + 2 \geq 0$. From the first, $x \geq 0$ and $x^2 - x - 2 \geq 0$, that is, $x \geq 2$. So the domain is the interval $[2, \infty[$. Squaring and rearranging:

$$x - \sqrt{x + 2} = 4, \quad x - 4 = \sqrt{x + 2}$$

A solution can only exist for $x \geq 4$. Squaring again and rearranging, $x^2 - 9x + 14 = 0$, so $x_1 = 7$ and $x_2 = 2$. The second root is less than 4, so it is not a solution. 7 is in the domain, and checking shows that the only solution of the equation is $x = 7$.

b) The domain of the equation is the set of positive real numbers. We bring the logarithms to the common base 2:

$$\log_2 x + \frac{\log_2 x}{\log_2 4} + \frac{\log_2 x}{\log_2 8} = \frac{\log_2 x^3}{\log_2 16}, \quad \text{that is,} \quad \log_2 x + \frac{\log_2 x}{2} + \frac{\log_2 x}{3} = \frac{3 \log_2 x}{4}$$

Multiplying by 12: $12 \log_2 x + 6 \log_2 x + 4 \log_2 x = 9 \log_2 x$, so $13 \log_2 x = 0$. Since $\log_2 x$ is strictly increasing and is zero at $x = 1$, the only solution is $x = 1$, which is in the domain.

3. Observing the habits of our family, the probability that we go for a walk on a summer afternoon is 0.6. If we stay at home, the probability that we eat ice cream is 0.25. The probability that we eat ice cream is 0.2. What is the probability that on a summer afternoon

- we stay at home and eat ice cream;
- we go for a walk and eat ice cream;
- we eat ice cream, given that we go for a walk?

(13 points)

Solution: Let S be the event that we go for a walk and F the event that we eat ice cream. Then $\bar{S}$ means staying at home.

a) We know that $P(S) = 0.6$, $P(F | \bar{S}) = 0.25$, $P(F) = 0.2$. So $P(\bar{S}) = 1 - P(S) = 0.4$. By the definition of conditional probability, $P(F | \bar{S}) = \frac{P(F\bar{S})}{P(\bar{S})}$, so $P(F\bar{S}) = 0.25 \cdot 0.4 = 0.1$. The probability that we stay at home and eat ice cream is 0.1.

b) Since $P(F) = P(F\bar{S}) + P(FS)$, we get $P(FS) = 0.2 - 0.1 = 0.1$.

c) $P(F | S) = \frac{P(FS)}{P(S)} = \frac{0.1}{0.6} = \frac{1}{6}$.

4. The function $f: x \mapsto x^3$ is given.

- Give the equations of the tangent lines to the graph of f whose slope is 12.
- Compute the distance between the points of tangency.
- Find the distance between the tangent lines.

(13 points)

Solution: a) $f'(x) = 3x^2$, which is 12 if $x = \pm 2$. So the points of tangency are $E_1(-2, -8)$ and $E_2(2, 8)$, and the tangent lines are $e_1: y = 12x + 16$ and $e_2: y = 12x - 16$.

b) The distance of the two points is $E_1E_2 = \sqrt{4^2 + 16^2} = 4\sqrt{17}$.

c) The distance between the two parallel lines equals the distance of the point $E_2(2, 8)$ from the line $e_1: 12x - y + 16 = 0$:

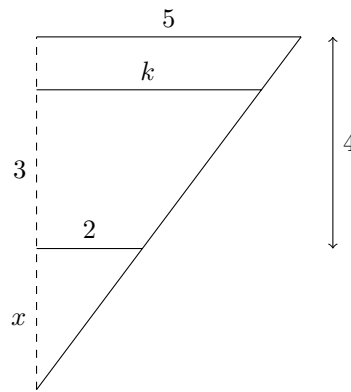
$$d = \frac{|12 \cdot 2 - 8 + 16|}{\sqrt{12^2 + (-1)^2}} = \frac{32}{\sqrt{145}} \approx 2.6575$$

Part II

5. A ball was put into a vessel whose inside has the shape of a frustum of a cone, and which was filled with water up to a height of 3 cm. After the ball was put in, the water just did not overflow. The radii of the bottom and top circles of the frustum are 2 cm and 5 cm, and its height is 4 cm. What is the radius of the ball, if it is immersed in the water up to $\frac{2}{3}$ of its height?

(16 points)

Solution: Consider the figure, which shows half of the cross-section of the vessel through its axis, extended to the apex of the cone.



By similarity, $\frac{2}{5} = \frac{x}{x+4}$, so $x = \frac{8}{3}$ cm; and $\frac{2}{k} = \frac{x}{x+3}$, so $k = 4.25$ cm, the radius of the water surface before the ball was put in. The volume of the frustum of height 1 cm with radii 5 cm and 4.25 cm is

$$V_{\text{frustum}} = \frac{\pi}{3} \cdot 1 \cdot (5^2 + 4.25^2 + 5 \cdot 4.25) = \frac{64.3125\pi}{3} \text{ cm}^3$$

This volume equals the volume of a spherical cap whose radius is the radius r in question and whose height is $m = \frac{2}{3} \cdot 2r = \frac{4}{3}r$, since the ball is immersed up to $\frac{2}{3}$ of its height:

$$V_{\text{cap}} = \frac{\pi}{3} m^2 (3r - m) = \frac{\pi}{3} \cdot \left(\frac{4}{3}r\right)^2 \cdot \left(3r - \frac{4}{3}r\right) = \frac{\pi}{3} \cdot \frac{80}{27} r^3 = \frac{64.3125\pi}{3}$$

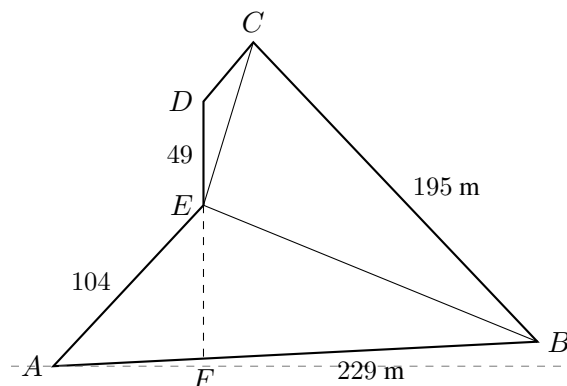
so $r \approx 2.7895$ cm. The radius of the ball is about 2.79 cm.

6. In the new building of the Újpest Technical School of the Wood and Metal Industry, teaching began in 1927. The school was built on a pentagonal lot, about which the local history newspaper writes: “Görgey Road and Szent Imre Street form an angle of 49.4° with each other, with straight lot lines of 229 m and 195 m. The lot line on Corvin Street breaks toward Görgey Road at an angle of 44.1° after 104 m, then takes a northern direction at a reflex angle for 49 m, and from there closes in a short section to the endpoint of the lot line on Szent Imre Street.” We also know that Görgey Road forms an angle of $+2.9^\circ$ with the east–west direction, and that the southwest corner of the lot is the intersection of Görgey Road and Corvin Street. Compute the area of the lot.

(16 points)

Remark: The wording of the problem was not unambiguous. The solution works with the real lot. Of course any other correct interpretation is acceptable.

Solution: We use the labels of the figure: A is the southwest corner, $AB = 229$ m along Görgey Road, $BC = 195$ m along Szent Imre Street, $AE = 104$ m and $ED = 49$ m along Corvin Street, and F is the point of AB due south of E .



First consider the triangle ABE . By the law of cosines,

$$EB^2 = 104^2 + 229^2 - 2 \cdot 104 \cdot 229 \cos 44.1^\circ, \quad \text{so} \quad EB \approx 170.44 \text{ m}$$

By the law of sines in the same triangle, $\frac{\sin \angle ABE}{\sin 44.1^\circ} = \frac{104}{170.44}$; since $\angle ABE < 49.4^\circ$, this gives $\angle ABE \approx 25.13^\circ$.

In the triangle EBC , $\angle CBE = \angle ABC - \angle ABE \approx 24.27^\circ$. By the law of cosines,

$$EC^2 = 170.44^2 + 195^2 - 2 \cdot 170.44 \cdot 195 \cos 24.27^\circ, \quad \text{so} \quad EC \approx 80.49 \text{ m}$$

By the law of cosines in the triangle EBC again, $195^2 = 80.49^2 + 170.44^2 - 2 \cdot 80.49 \cdot 170.44 \cos \angle CEB$, so $\cos \angle CEB \approx -0.0910$ and $\angle CEB \approx 95.22^\circ$.

Since $\angle EFB = 90^\circ - 2.9^\circ = 87.1^\circ$, we get $\angle BEF \approx 180^\circ - 25.13^\circ - 87.1^\circ = 67.77^\circ$. So $\angle CED = 180^\circ - \angle BEF - \angle CEB \approx 17.01^\circ$. The area of the lot is

$$\begin{aligned} T &= t_{ABE} + t_{EBC} + t_{ECD} \\ &\approx \frac{1}{2} (104 \cdot 229 \sin 44.1^\circ + 195 \cdot 170.44 \sin 24.27^\circ + 49 \cdot 80.49 \sin 17.01^\circ) \approx 15\,695 \text{ m}^2 \end{aligned}$$

7. Consider the complete graph on 6 vertices.

- How many cycles with 3, 4, 5, and 6 vertices are there in this graph?
- How many cycles with 3, 4, 5, and 6 vertices will there be, if we delete one edge?

(16 points)

Solution: a) 3 vertices: $\binom{6}{3} = 20$, since 3 vertices determine only one cycle. 4 vertices: $\binom{6}{4} \cdot \frac{4!}{4 \cdot 2} = 15 \cdot 3 = 45$, since for 4 vertices the order of the vertices can be chosen in $4!$ ways, but it does not matter which vertex is the first (we divide by 4), and a given order and the reverse order give the same cycle (we divide by 2). By the same reasoning, the number of cycles with 5 vertices is $\binom{6}{5} \cdot \frac{5!}{5 \cdot 2} = 6 \cdot 12 = 72$, and with 6 vertices $\binom{6}{6} \cdot \frac{6!}{6 \cdot 2} = 60$.

b) We count how many fewer cycles there are in each case because of deleting the edge: these are the cycles one of whose edges is the chosen edge, with the other vertices chosen from the other four vertices. 3 vertices: we choose 1 of the remaining 4 vertices, $\binom{4}{1} = 4$ possibilities. 4 vertices: $\binom{4}{2} \cdot 2! = 12$ cycles (the two chosen vertices can be attached to the endpoints of the given edge in $2!$ orders). 5 vertices: $\binom{4}{3} \cdot 3! = 24$ cycles, and 6 vertices: $\binom{4}{4} \cdot 4! = 24$ cycles.

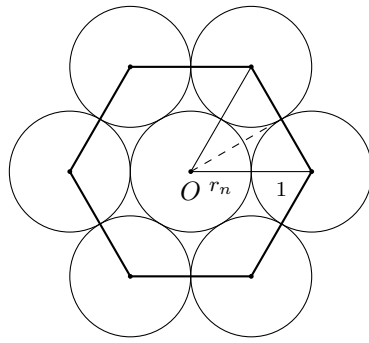
So the numbers of cycles are $20 - 4 = 16$, $45 - 12 = 33$, $72 - 24 = 48$, and $60 - 24 = 36$.

8. About each vertex of a regular n -gon with sides of length 2 ($n \geq 3$), as center, we draw a circle of unit radius.

- Give, as a function of n , the radius r_n of the circle that touches each of the circles drawn about the vertices of the polygon, inside the polygon.
- For which values of n will r_n be greater than 2013?

(16 points)

Solution: The segment joining the center O of the circle to a vertex of the polygon has length equal to the sum of the radii of the two circles, that is, $r_n + 1$.



a) Two neighboring circles about the vertices touch each other at the midpoint of the side. In the right triangle formed by O , a vertex, and the midpoint of a side adjacent to it, the angle at O is $\frac{\pi}{n}$ and the side opposite it is 1 (half the side), so $\frac{1}{r_n+1} = \sin \frac{\pi}{n}$, and

$$r_n = \frac{1}{\sin \frac{\pi}{n}} - 1$$

(The figure shows the case $n = 6$, where $r_6 = 1$.)

b)

$$\frac{1}{\sin \frac{\pi}{n}} - 1 > 2013, \quad \frac{1}{\sin \frac{\pi}{n}} > 2014, \quad \sin \frac{\pi}{n} < \frac{1}{2014}$$

Since $\frac{\pi}{n} \in]0, \frac{\pi}{3}]$, where the sine function is increasing, $\frac{\pi}{n} < 0.00049652$, so $n > \frac{\pi}{0.00049652} \approx 6327.2$. So r_n is greater than 2013 for $n \geq 6328$.

9. A university student in Bergengocia took out a student loan during the 3 years of studies. From September of the first academic year, the student received 25 000 Ft on the first day of every month. Interest is computed monthly by proportional (linear) interest, that is, the bank computes the monthly interest by the formula $\frac{t \cdot k}{12}$, where t is the principal owed and k is the current annual interest rate. Interest not yet paid in a given year is capitalized every year on December 31. Suppose that the interest rate is 9% during the whole time the loan is taken out and repaid.

- a) What is the principal owed by the graduate on the first day of the month (July) after finishing the studies?
- b) From September 1 after finishing the studies until the debt is fully paid, the graduate repays 25 000 Ft each month (or the remaining amount in the last month). The amount repaid each month first reduces only the accumulated interest, and after that is used up, it reduces the principal each month as well. How many months does it take to repay the whole loan this way?

(16 points)

Solution: a) On the last day of the first calendar year the principal is 100 000 Ft. The interest accumulated during this period is

$$(4 + 3 + 2 + 1) \cdot \frac{0.09}{12} \cdot 25\,000 = 1875 \text{ Ft}$$

The next calendar year the principal starts from 101 875 Ft, and on the last day of the year it is 300 000 Ft more, 401 875 Ft. The interest charged in this year is

$$(12 + 11 + \cdots + 1) \cdot \frac{0.09}{12} \cdot 25\,000 + 101\,875 \cdot 0.09 \approx 23\,794 \text{ Ft}$$

At the beginning of the third calendar year the principal is 425 669 Ft, and at the end of the year 725 669 Ft. The interest accumulated in this year is 52 935 Ft. At the beginning of the fourth calendar year the principal is 778 604 Ft, and on July 1 it is 928 604 Ft.

- b) The interest accumulated in the first half of the fourth calendar year is 38 975 Ft. In the next two months another 13 929 Ft of interest is added, 52 904 Ft altogether.

In the next three months the interest is always about 6965 Ft, and the repayment is 25 000 Ft. So by the end of November the graduate has paid the interest accumulated until then, and has also repaid about 1202 Ft of the principal. On December 1 the principal is 927 402 Ft. After this, every month the graduate pays the interest and also repays some of the principal. After n more months the principal is

$$\left(\dots \left\{ \left[927\,402 \left(1 + \frac{0.09}{12} \right) - 25\,000 \right] \left(1 + \frac{0.09}{12} \right) - 25\,000 \right\} \dots \right)$$

The graduate has repaid the loan when this value is not positive:

$$927\,402 \cdot 1.0075^n - 25\,000 \cdot (1.0075^{n-1} + 1.0075^{n-2} + \dots + 1.0075 + 1) \leq 0$$

Using the formula for the sum of a geometric sequence:

$$\begin{aligned} 927\,402 \cdot 1.0075^n - 25\,000 \cdot \frac{1.0075^n - 1}{1.0075 - 1} &\leq 0 \\ 0.278\,221 \cdot 1.0075^n - (1.0075^n - 1) &\leq 0 \\ -0.721\,779 \cdot 1.0075^n + 1 &\leq 0 \\ 1.3855 &\leq 1.0075^n, \quad \text{so} \quad n \geq \frac{\lg 1.3855}{\lg 1.0075} \approx 43.63 \end{aligned}$$

So the graduate pays 25 000 Ft for $3 + 43 = 46$ months and a smaller amount (about 15 900 Ft) in the last month: the whole repayment takes 47 months.

Note: KoMaL writes $-0.727\,794 \cdot 1.0075^n + 1 \leq 0$ instead of $1 - 0.278\,221 = 0.721\,779$, gets $n \geq 42.52$ and 46 months. A month-by-month computation confirms the 47 months. (KoMaL also prints 13 939 instead of 13 929 for the interest of July and August; its total 52 904 is correct.)