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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. In the triangle ABC inscribed in a circle of radius r , $AB = r$ and $AC = r\sqrt{3}$. How long is the side BC ?

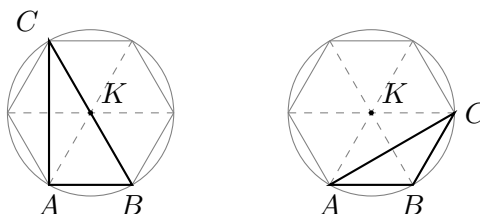
(11 points)

Solution: Let K be the center of the circle of radius r , and let x be the length of the side BC . The triangle AKB is equilateral, so the central angle AKB belonging to the (smaller) arc AB is 60° . So the interior angle of the triangle ABC at the vertex C is 30° , since it is an inscribed angle belonging to that arc. By the law of cosines in the triangle ABC :

$$\begin{aligned} r^2 &= x^2 + (r\sqrt{3})^2 - 2\sqrt{3} \cdot r \cdot x \cdot \cos 30^\circ \\ 0 &= x^2 - 3rx + 2r^2 \end{aligned}$$

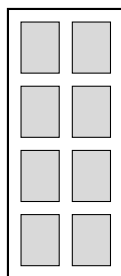
Solving this quadratic equation in x :

$$x_{1,2} = \frac{3r \pm \sqrt{9r^2 - 8r^2}}{2} = \frac{3r \pm r}{2}, \quad \text{so} \quad x_1 = 2r, \quad x_2 = r$$



So the possible lengths of the side BC are $2r$ and r .

2. The figure shows a sketch of a paneled door 90 cm wide and 210 cm tall.



The eight identical rectangular panels take up exactly half of the door leaf. The strips between and beside the panels have the same width everywhere. What is this width?

(13 points)

Solution: Let the width of the strips be x . Cutting the part outside the panels into five horizontal strips and three vertical pieces, its area equals half the area of the door leaf:

$$\begin{aligned} 5 \cdot 90 \cdot x + 3 \cdot x \cdot (210 - 5x) &= 9450 \\ 450x + 630x - 15x^2 &= 9450 \\ x^2 - 72x + 630 &= 0 \end{aligned}$$

By the quadratic formula, $x_{1,2} = \frac{72 \pm \sqrt{2664}}{2}$, that is, $x_1 \approx 10.2$ and $x_2 \approx 61.8$. Since the door is 90 cm wide, the strips are narrower than 30 cm, so only x_1 can be the answer. So the strips around the panels on the door are about 10.2 cm wide.

3. Solve the following equation.

$$(x - 5 - 2\sqrt{x - 2})(x - 10 - 4\sqrt{x - 5}) = 0$$

(13 points)

Solution: The domain of the equation is $x \geq 5$. The equation can be written in the form

$$\begin{aligned} (x - 2 - 2\sqrt{x - 2} - 3)(x - 5 - 4\sqrt{x - 5} - 5) &= 0 \\ [(\sqrt{x - 2})^2 - 2\sqrt{x - 2} - 3] \cdot [(\sqrt{x - 5})^2 - 4\sqrt{x - 5} - 5] &= 0 \end{aligned}$$

The first factor is quadratic in $\sqrt{x - 2}$, and the second in $\sqrt{x - 5}$. Factoring these further:

$$(\sqrt{x - 2} - 3)(\sqrt{x - 2} + 1)(\sqrt{x - 5} - 5)(\sqrt{x - 5} + 1) = 0$$

A product is zero if at least one of its factors is zero. The second and the fourth factors cannot be less than 1, so they are never zero. Two cases remain.

Case I: $\sqrt{x - 2} - 3 = 0$, so $x = 11$.

Case II: $\sqrt{x - 5} - 5 = 0$, so $x = 30$.

Both are solutions, since they are in the domain. (It is also easy to check the roots.)

4. The function $f:]-3, 5[\rightarrow \mathbb{R}$, $f(x) = \left| \left| (x - 1)^2 - 1 \right| - 3 \right| - 5$ is given.

- Find the zeros of the function.
- Give the coordinates of the lattice points that lie on the graph of the function.
- On which intervals is the function strictly decreasing?

(14 points)

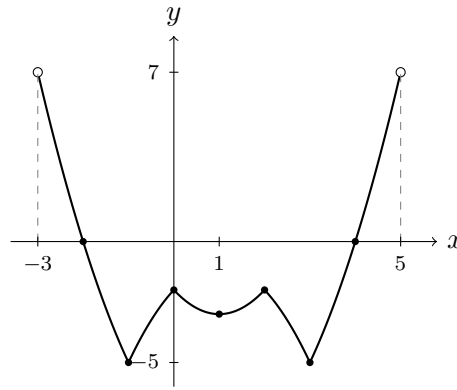
Solution: a) We look for the solutions of $\left| \left| (x - 1)^2 - 1 \right| - 3 \right| - 5 = 0$. The value of $\left| (x - 1)^2 - 1 \right| - 3$ can be -5 or 5 , that is,

$$\left| (x - 1)^2 - 1 \right| = -2 \quad \text{or} \quad \left| (x - 1)^2 - 1 \right| = 8$$

The first equation has no real solution, so only $(x - 1)^2 = 9$ can give a solution (and $(x - 1)^2 = -7$ is impossible). So $x_1 = -2$, $x_2 = 4$.

b) There are seven integers in the domain of the function: $-2, -1, 0, 1, 2, 3$, and 4 . So at most seven lattice points can lie on the graph. We compute the values of the function at these places; since each is an integer, the lattice points are

$$(-2, 0), (-1, -5), (0, -2), (1, -3), (2, -2), (3, -5), (4, 0)$$



c) The graph of the function can be drawn by transforming the graph of $y = x^2$. The function is strictly decreasing on the intervals $]-3, -1]$, $[0, 1]$, and $[2, 3]$.

Part II

5. About the positive digits a , b , and c we know that $a + b + c = \overline{ab}$ and $a^2 + b^2 + c^2 = \overline{b^2a}$ (where $\overline{ab}$ and $\overline{b^2a}$ are two-digit numbers). Find these digits.

(16 points)

Solution: The problem shows that b is a positive digit whose square is also a digit. So there are three cases.

Case I: If $b = 3$, we must solve the system $a + 3 + c = 10a + 3$, $a^2 + 9 + c^2 = 90 + a$, which can be written as

$$c = 9a, \quad a^2 + c^2 = a + 81$$

a is a positive digit whose 9-fold is also a digit, so $a = 1$, $c = 9$. This is indeed a solution, since it satisfies the second equation too.

Case II: If $b = 2$, the system is $a + 2 + c = 10a + 2$, $a^2 + 4 + c^2 = 40 + a$, that is, $c = 9a$, $a^2 + c^2 = a + 36$. Again only $a = 1$, $c = 9$ would follow from the first equation, but this is not a solution of the second.

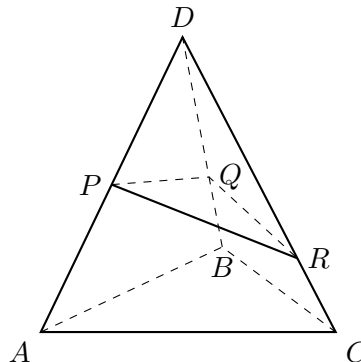
Case III: If $b = 1$, the system is $a + 1 + c = 10a + 1$, $a^2 + 1 + c^2 = 10 + a$, that is, $c = 9a$, $a^2 + c^2 = a + 9$. As in the previous case, we get no suitable digits.

The only solution of the problem is $a = 1$, $b = 3$, $c = 9$.

6. The regular tetrahedron $ABCD$ is cut by a plane. The cutting plane cuts the three edges starting from one vertex in the ratios $1 : 1$, $2 : 1$, and $3 : 1$, measured from the common vertex. Find the ratio of the surface areas of the tetrahedron cut off and the original tetrahedron.

(16 points)

Solution: We use the labels of the figure. We know that $DP : PA = 1 : 1$, $DQ : QB = 2 : 1$, $DR : RC = 3 : 1$. Let the edge of the regular tetrahedron be a ; then $PD = \frac{a}{2}$, $QD = \frac{2a}{3}$, $RD = \frac{3a}{4}$.



The surface area of the regular tetrahedron $ABCD$ is $A_{ABCD} = a^2\sqrt{3}$. Each lateral face of the tetrahedron cut off has a 60° angle, so the areas of these three triangles are

$$\begin{aligned} t_{PQD} &= \frac{\frac{a}{2} \cdot \frac{2a}{3} \cdot \sin 60^\circ}{2} = \frac{a^2\sqrt{3}}{12} \approx 0.144a^2 \\ t_{QRD} &= \frac{\frac{2a}{3} \cdot \frac{3a}{4} \cdot \sin 60^\circ}{2} = \frac{a^2\sqrt{3}}{8} \approx 0.217a^2 \\ t_{RPD} &= \frac{\frac{3a}{4} \cdot \frac{a}{2} \cdot \sin 60^\circ}{2} = \frac{3a^2\sqrt{3}}{32} \approx 0.162a^2 \end{aligned}$$

Each side of the bottom face of the tetrahedron cut off can be found by the law of cosines:

$$\begin{aligned} PQ^2 &= \left(\frac{a}{2}\right)^2 + \left(\frac{2a}{3}\right)^2 - 2 \cdot \frac{a}{2} \cdot \frac{2a}{3} \cos 60^\circ = \frac{13a^2}{36}, & PQ &= \frac{a\sqrt{13}}{6} \approx 0.601a \\ QR^2 &= \left(\frac{2a}{3}\right)^2 + \left(\frac{3a}{4}\right)^2 - 2 \cdot \frac{2a}{3} \cdot \frac{3a}{4} \cos 60^\circ = \frac{73a^2}{144}, & QR &= \frac{a\sqrt{73}}{12} \approx 0.712a \\ RP^2 &= \left(\frac{3a}{4}\right)^2 + \left(\frac{a}{2}\right)^2 - 2 \cdot \frac{3a}{4} \cdot \frac{a}{2} \cos 60^\circ = \frac{7a^2}{16}, & RP &= \frac{a\sqrt{7}}{4} \approx 0.661a \end{aligned}$$

We find the area of the triangle PQR from its sides by Heron's formula. Half its perimeter is $s \approx \frac{0.601a + 0.712a + 0.661a}{2} = 0.987a$, and

$$t_{PQR} = \sqrt{s(s-PQ)(s-QR)(s-RP)} \approx \sqrt{0.987a \cdot 0.386a \cdot 0.275a \cdot 0.326a} \approx 0.185a^2$$

The surface area of the tetrahedron cut off is the sum of the areas of the four triangles: $A_{PQRD} \approx 0.144a^2 + 0.217a^2 + 0.162a^2 + 0.185a^2 = 0.708a^2$. The ratio of the surface areas is

$$\frac{A_{PQRD}}{A_{ABCD}} \approx \frac{0.708a^2}{\sqrt{3}a^2} \approx 0.41$$

7. Solve the equation

$$\sin^4 x + \sin^4 \left(x + \frac{\pi}{4}\right) + \sin^4 \left(x - \frac{\pi}{4}\right) = \frac{3}{2}$$

over the real numbers.

(16 points)

Solution: Multiplying both sides by 4, the equation can be written as

$$(2\sin^2 x)^2 + \left[2\sin^2 \left(x + \frac{\pi}{4}\right)\right]^2 + \left[2\sin^2 \left(x - \frac{\pi}{4}\right)\right]^2 = 6$$

We know that $2\sin^2 x = (\cos^2 x + \sin^2 x) - (\cos^2 x - \sin^2 x) = 1 - \cos 2x$. Using this,

$$2\sin^2 \left(x + \frac{\pi}{4}\right) = 1 - \cos \left(2x + \frac{\pi}{2}\right) = 1 + \sin 2x, \quad 2\sin^2 \left(x - \frac{\pi}{4}\right) = 1 - \cos \left(2x - \frac{\pi}{2}\right) = 1 - \sin 2x$$

Substituting these into the equation:

$$\begin{aligned} (1 - \cos 2x)^2 + (1 + \sin 2x)^2 + (1 - \sin 2x)^2 &= 6 \\ 3 - 2\cos 2x + \cos^2 2x + 2\sin^2 2x &= 6 \\ \sin^2 2x - 2\cos 2x + 4 &= 6 \end{aligned}$$

(using $\cos^2 2x + \sin^2 2x = 1$). Transforming further, we get a quadratic equation:

$$1 - \cos^2 2x - 2\cos 2x + 4 = 6, \quad \cos^2 2x + 2\cos 2x + 1 = 0, \quad (\cos 2x + 1)^2 = 0$$

So $\cos 2x = -1$, which gives $2x = \pi + 2k\pi$ ($k \in \mathbb{Z}$). The solutions of the equation are $x = \frac{\pi}{2} + k\pi$ ($k \in \mathbb{Z}$).

8. Consider the parabolas $y = (p - 1)x^2 + 2x - (p + 1)$, where p is a real parameter and $p \neq 1$.
- Which of these parabolas cut the x -axis at two points with integer coordinates, or touch it at a point with integer coordinates?
 - Write the equation of the tangent line to the parabola with $p = 2$ at its point with first coordinate 4.
 - Find the area under the parabola with $p = 3$ on the interval $[1, 3]$.

(16 points)

Solution: a) We need the zeros of the quadratic equation $(p - 1)x^2 + 2x - (p + 1) = 0$ with parameter:

$$x_{1,2} = \frac{-2 \pm \sqrt{4 + 4(p - 1)(p + 1)}}{2(p - 1)} = \frac{-2 \pm 2p}{2(p - 1)} = \frac{-1 \pm p}{p - 1}$$

So the zeros are $x_1 = 1$ and $x_2 = \frac{-p-1}{p-1}$. (Of course for $p = 0$, $x_1 = x_2$.) We only have to examine for which parameters x_2 is an integer, since x_1 is always an integer:

$$x_2 = \frac{-p-1}{p-1} = \frac{-(p-1)-2}{p-1} = -1 - \frac{2}{p-1}$$

$p - 1$ must be a divisor of 2, so its possible values are $-2, -1, 1, 2$. This gives four suitable values of p : $-1, 0, 2, 3$. The parabolas satisfying the conditions are

$$y = -2x^2 + 2x, \quad y = -x^2 + 2x - 1, \quad y = x^2 + 2x - 3, \quad y = 2x^2 + 2x - 4$$

b) For $p = 2$ the equation of the parabola is $y = x^2 + 2x - 3$. The slope of the tangent at each point is given by the derivative: $(x^2 + 2x - 3)' = 2x + 2$. So at the point with first coordinate 4, the slope of the tangent is $2 \cdot 4 + 2 = 10$. The second coordinate belonging to 4 is $4^2 + 2 \cdot 4 - 3 = 21$. So the tangent is the line through $(4, 21)$ with slope 10:

$$y - 21 = 10(x - 4), \quad \text{that is,} \quad y = 10x - 19$$

c) For $p = 3$ the equation of the parabola is $y = 2x^2 + 2x - 4$; its zeros are -2 and 1 , so it is nonnegative on $[1, 3]$. The area under the curve is given by the definite integral

$$\int_1^3 (2x^2 + 2x - 4) dx = \left[\frac{2x^3}{3} + x^2 - 4x \right]_1^3 = (18 + 9 - 12) - \left(\frac{2}{3} + 1 - 4 \right) = \frac{52}{3}$$

So for $p = 3$ the area under the parabola on the interval $[1, 3]$ is $\frac{52}{3}$.

9. For three different right circular cones we know that the radii of their base circles, and also the slant heights of the cones, are three consecutive terms of arithmetic sequences with the same common difference. Show that the surface areas of the cones cannot be three consecutive terms of an arithmetic sequence.

(16 points)

Solution: Let the radii of the base circles be $r - d, r, r + d$, and the slant heights $a - d, a, a + d$. Then the surface areas of the three cones are

$$A_1 = \pi(r - d)(a - d + r - d), \quad A_2 = \pi r(a + r), \quad A_3 = \pi(r + d)(a + d + r + d)$$

If A_1, A_2, A_3 were three consecutive terms of an arithmetic sequence, then $A_3 - A_2 = A_2 - A_1$ would have to hold. We compute these differences:

$$\begin{aligned} A_3 - A_2 &= \pi(r + d)(a + r + 2d) - \pi r(a + r) = \pi(ad + 3rd + 2d^2) \\ A_2 - A_1 &= \pi r(a + r) - \pi(r - d)(a + r - 2d) = \pi(ad + 3rd - 2d^2) \end{aligned}$$

The two differences could be equal only if $d = 0$, but the cones are different, so $d \neq 0$. So the surface areas of the cones cannot be three consecutive terms of an arithmetic sequence.