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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. We wrote the differences of neighboring terms of a geometric sequence (with positive terms) one after the other. Prove that the sequence obtained this way is also a geometric sequence, if $q \neq 1$. Give the first term and the common ratio of the geometric sequence obtained this way.

(11 points)

Solution: Let the first term of the original geometric sequence be a_1 and its common ratio q . Then its first few terms are $a_1, a_1q, a_1q^2, a_1q^3, \dots$. The differences of neighboring terms, in order, are

$$a_1q - a_1, a_1q^2 - a_1q, a_1q^3 - a_1q^2, \dots, a_1q^n - a_1q^{n-1}, \dots$$

Factoring out $a_1(q-1)$ and the powers of q :

$$a_1(q-1), a_1q(q-1), a_1q^2(q-1), \dots, a_1q^{n-1}(q-1), \dots$$

So the sequence obtained is indeed a geometric sequence (its terms are nonzero, since $q \neq 1$). Its first term is $a_1(q-1)$, and its common ratio is q .

2. a) Réka sells environmentally friendly laundry gel in 3 liter packages. She put forty-five colored slips into a small basket. Of these slips, eighteen say “one gift”, fifteen “two gifts”, and twelve “three gifts”. Before buying, customers draw a slip, and according to the text on it they get gifts in the following order: a 810 Ft all-purpose cleaner, a 510 Ft fabric softener, a 290 Ft dishwashing liquid. Then they put the slip back into the basket.

Before buying, Laszlo also draws a slip. What is the probability that he will get three gifts?

What should the price of one liter of the laundry gel be, if the distributor wants an average revenue of 535 Ft per liter?

- b) We forgot the four-digit code needed to turn on the phone. We only know that it consisted of four consecutive positive digits, but the digits follow each other in neither increasing nor decreasing order. How many such four-digit numbers are there?

(13 points)

Solution: a) There are forty-five slips in the basket, of which only twelve are favorable. We may assume that every slip is drawn with the same probability, so we can use the classical model of probability:

$$P(\text{three gifts}) = \frac{12}{45} = \frac{4}{15} \approx 0.27$$

Let the price of one liter of laundry gel be x Ft; then a 3 liter package costs $3x$ Ft. Every customer draws a slip, so everybody gets gifts whose value has to be subtracted from the price of the package. We have

$$P(\text{three gifts}) = \frac{12}{45} = \frac{4}{15}, \quad P(\text{two gifts}) = \frac{15}{45} = \frac{1}{3}, \quad P(\text{one gift}) = \frac{18}{45} = \frac{2}{5}$$

The distributor wants an average revenue of $3 \cdot 535 = 1605$ Ft per 3 liter package. A customer pays, after the value of the gifts, $3x - 810 - 510 - 290$ Ft with probability $\frac{4}{15}$, $3x - 810 - 510$ Ft with probability $\frac{1}{3}$, and $3x - 810$ Ft with probability $\frac{2}{5}$. So

$$\begin{aligned}\frac{4}{15}(3x - 1610) + \frac{1}{3}(3x - 1320) + \frac{2}{5}(3x - 810) &= 1605 \\ 4(3x - 1610) + 5(3x - 1320) + 6(3x - 810) &= 24075 \\ 45x - 17900 &= 24075, \quad x \approx 932.8\end{aligned}$$

So the price of one liter of laundry gel should be about 933 Ft (and then the 3 liter package can be priced at 2799 Ft).

b) The possible sets of four digits are $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$, $\{3, 4, 5, 6\}$, $\{4, 5, 6, 7\}$, $\{5, 6, 7, 8\}$, $\{6, 7, 8, 9\}$. From each of them $4! = 24$ four-digit numbers can be formed, but two of these are not suitable (the increasing and the decreasing order). So there are $6 \cdot 22 = 132$ suitable four-digit numbers.

3. At each vertex of a regular hexagon $ABCDEF$ with side length 6 units, lying in a horizontal plane, a segment stands perpendicular to the plane of the hexagon, all on the same side of the plane. The lengths of these segments are

$$AA' = 2, \quad BB' = \frac{9}{2}, \quad CC' = 7, \quad DD' = \frac{19}{2}, \quad EE' = 12, \quad FF' = \frac{29}{2}$$

- a) How long is the closed broken line $A'B'C'D'E'F'$?
b) Find the pairs of points of $A'D'$, $B'E'$, $C'F'$ that lie exactly above each other, and find the distances between these pairs of points.

(13 points)

Solution: a) The heights of neighboring segments differ by $\frac{5}{2}$, except for AA' and FF' . So five pieces of the broken line are hypotenuses of congruent right triangles with legs 6 and $\frac{5}{2}$:

$$A'B' = \sqrt{6^2 + \left(\frac{5}{2}\right)^2} = \frac{13}{2}$$

The missing piece, $A'F'$, is also the hypotenuse of a right triangle, with legs 6 and $\frac{29}{2} - 2 = \frac{25}{2}$:

$$A'F' = \sqrt{6^2 + \left(\frac{25}{2}\right)^2} = \sqrt{36 + \frac{625}{4}} = \frac{\sqrt{769}}{2}$$

So the length of the broken line is

$$5 \cdot \frac{13}{2} + \frac{\sqrt{769}}{2} = \frac{65 + \sqrt{769}}{2} \approx 46.37$$

b) The segments $A'D'$, $B'E'$, $C'F'$ lie above the long diagonals AD , BE , CF , which meet only at the center K of the hexagon. So the points in question are the midpoints P_1 , P_2 , P_3 of $A'D'$, $B'E'$, $C'F'$, which all lie above K . Their heights are the midlines of the trapezoids $ADD'A'$, $BEE'B'$, $CFE'F'$:

$$KP_1 = \frac{2 + \frac{19}{2}}{2} = \frac{23}{4}, \quad KP_2 = \frac{\frac{9}{2} + 12}{2} = \frac{33}{4}, \quad KP_3 = \frac{7 + \frac{29}{2}}{2} = \frac{43}{4}$$

So P_3 is the highest, P_2 is $\frac{5}{2}$ below it, and P_1 is another $\frac{5}{2}$ below that: $P_1P_2 = P_2P_3 = \frac{5}{2}$ and $P_1P_3 = 5$.

4. Consider the sequence given by its general term

$$a_n = a \cdot n^2 + (2a + b) \cdot n - (b^2 - b - a)$$

Show that if a and b are integers, then the sum of any five consecutive terms of the sequence is divisible by 5.

(14 points)

Solution: Let the sum of five consecutive terms be $a_{n-2} + a_{n-1} + a_n + a_{n+1} + a_{n+2}$. Since

$$(n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 = 5n^2 + 10 \quad \text{and} \quad (n-2) + (n-1) + n + (n+1) + (n+2) = 5n$$

the sum is

$$(5n^2 + 10)a + 5n(2a + b) - 5(b^2 - b - a) = 5(an^2 + 2a + 2an + bn - b^2 + b + a)$$

The expression in parentheses is clearly an integer. So the sum of any five consecutive terms of the sequence is indeed divisible by 5.

Part II

5. Solve the following equations.

a) $2^{3x} - 2^{2x} - 2^{x+1} = 0$

b) $\sin 3x - \sin 2x - \sin x = 0$

(16 points)

Solution: a) Factoring out 2^x on the left side, the equation can be written as

$$2^x \left[(2^x)^2 - 2^x - 2 \right] = 2^x (2^x - 2) (2^x + 1) = 0$$

So there are three possibilities for the value of 2^x : 0, 2, and -1 . From the range of the exponential function 2^x , only 2 is possible, and then $x = 1$. The only solution of the equation is 1.

b) We use the addition formulas:

$$\begin{aligned} (\sin 2x \cos x + \cos 2x \sin x) - 2 \sin x \cos x - \sin x &= 0 \\ 2 \sin x \cos^2 x + (\cos^2 x - \sin^2 x) \sin x - 2 \sin x \cos x - \sin x &= 0 \end{aligned}$$

Factoring out $\sin x$ and using $\sin^2 x = 1 - \cos^2 x$:

$$\begin{aligned} \sin x (3 \cos^2 x - \sin^2 x - 2 \cos x - 1) &= 0 \\ \sin x (4 \cos^2 x - 2 \cos x - 2) &= 0 \\ \sin x (2 \cos^2 x - \cos x - 1) &= 0 \end{aligned}$$

A product is zero if at least one of its factors is zero. We get two cases.

I. $\sin x = 0$. Then $x = k\pi$, $k \in \mathbb{Z}$.

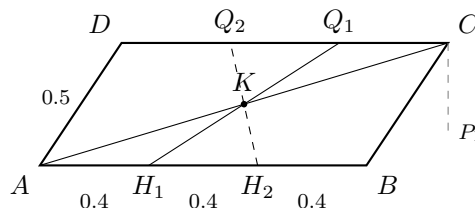
II. $2 \cos^2 x - \cos x - 1 = 0$, so $\cos x = 1$ or $\cos x = -\frac{1}{2}$. The first gives $x = 2k\pi$ (already among the solutions of case I); the second gives $x = \pm \frac{2\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$.

So the solutions are $x = k\pi$ and $x = \pm \frac{2\pi}{3} + 2k\pi$ ($k \in \mathbb{Z}$); more briefly, $x = k \cdot \frac{2\pi}{3}$ or $x = \pi + 2k\pi$, $k \in \mathbb{Z}$.

6. The longer edge of a board 0.5 meters wide and 1.2 meters long rests on the horizontal floor, and the edge parallel to it is at a height of 30 cm.
- What is the angle of elevation of an ant going up along a diagonal of the board?
 - How does this angle change, if the ant goes from one of the points dividing the lower edge into three equal parts to the non-opposite point dividing the upper edge into three equal parts?
 - Find the angle between the two paths.

(16 points)

Solution: The figure shows the board $ABCD$; lengths are in meters. AB lies on the floor, CD is at height 0.3; H_1 , H_2 divide AB , and Q_1 , Q_2 divide DC into three equal parts.



Let the ant go up along the diagonal AC ; then the second path can be H_1Q_1 or H_2Q_2 . Of course these segments make the same angle with the horizontal, which only matters in part c), where two kinds of paths are possible.

- By the Pythagorean theorem in the right triangle ABC , $AC = 1.3$. The angle of elevation is the acute angle α at A of the right triangle AP_1C , where P_1 is the projection of C onto the floor: $\sin \alpha = \frac{0.3}{1.3}$, so $\alpha \approx 13.34^\circ$.
- By the Pythagorean theorem in the right triangle formed by H_1 , Q_1 and the point of the upper edge opposite H_1 , $H_1Q_1 = \sqrt{0.4^2 + 0.5^2} = \sqrt{0.41}$. So for the angle of elevation β , $\sin \beta = \frac{0.3}{\sqrt{0.41}}$, so $\beta \approx 27.94^\circ$. The segment H_2Q_2 gives the same angle. So the angle is 14.6° larger.
- We can find the angle, for example, by the law of cosines. Since the ant's second path can be of two kinds compared with the first, as in part b) we must consider two cases. The two paths meet at the point K , and

$$AK = \frac{AC}{2} = 0.65, \quad H_1K = H_2K = \frac{H_1Q_1}{2} = \sqrt{0.1025}$$

Case I: The angle γ at K of the triangle AKH_1 is one possibility for the angle in question. By the law of cosines,

$$0.4^2 = 0.65^2 + 0.1025 - 2 \cdot 0.65 \cdot \sqrt{0.1025} \cos \gamma, \quad \text{so} \quad \cos \gamma \approx 0.8770, \quad \gamma \approx 28.72^\circ$$

Case II: The angle δ at K of the triangle AKH_2 is the other possibility (or its supplement, if δ is obtuse):

$$0.8^2 = 0.65^2 + 0.1025 - 2 \cdot 0.65 \cdot \sqrt{0.1025} \cos \delta, \quad \text{so} \quad \cos \delta \approx -0.2763, \quad \delta \approx 106.04^\circ$$

whose supplement is $180^\circ - \delta \approx 73.96^\circ$. So the angle between the two paths is, rounded, either 28.72° or 73.96° .

7. Solve the following equation.

$$x - 8 - \frac{560}{x - 32} = 19 \cdot \sqrt{\frac{x - 8}{x - 32}}$$

(16 points)

Solution: The domain of the equation is given by $\frac{x-8}{x-32} \geq 0$, that is, $x \leq 8$ or $x > 32$. Multiplying both sides of the equation by $x - 32$ and rearranging:

$$(x - 8)(x - 32) - 19(x - 32)\sqrt{\frac{x - 8}{x - 32}} - 560 = 0$$

Case I: If $x \leq 8$, then $x - 32 < 0$, so $(x - 32)\sqrt{\frac{x-8}{x-32}} = -\sqrt{(x-8)(x-32)}$, and the equation can be written as

$$(x - 8)(x - 32) + 19\sqrt{(x - 8)(x - 32)} - 560 = 0$$

This is a quadratic equation in $\sqrt{(x - 8)(x - 32)}$, with roots $\frac{-19 \pm 51}{2}$, that is, 16 and -35 . Only the positive one is possible:

$$(x - 8)(x - 32) = 256, \quad x^2 - 40x = 0, \quad x_1 = 0, \quad x_2 = 40$$

Only $x_1 = 0$ is in the interval examined.

Case II: If $x > 32$, then $x - 32 > 0$, and the equation can be written as

$$(x - 8)(x - 32) - 19\sqrt{(x - 8)(x - 32)} - 560 = 0$$

The roots of this quadratic equation are $\frac{19 \pm 51}{2}$, that is, 35 and -16 . Again only the positive one is possible:

$$(x - 8)(x - 32) = 1225, \quad x^2 - 40x - 969 = 0, \quad x_3 = 57, \quad x_4 = -17$$

Only $x_3 = 57$ is in the interval examined.

So the equation has two solutions, 0 and 57. (Checking shows that they are indeed solutions of the equation.)

8. For the triangle with vertices $A(1, 2)$, $B(12, 4)$, and $C(4, 8)$, find

- its centroid;
- its largest angle;
- the value of $CP^2 + CQ^2$, where P is the intersection of the interior angle bisector from the vertex C with the line AB , and Q is the intersection of the exterior angle bisector from the vertex C with the line AB .

(16 points)

Solution: a) The coordinates of the centroid are the means of the corresponding coordinates of the vertices:

$$S\left(\frac{1 + 12 + 4}{3}, \frac{2 + 4 + 8}{3}\right) = S\left(\frac{17}{3}, \frac{14}{3}\right)$$

b) The lengths of the sides are

$$AB = \sqrt{(12 - 1)^2 + (4 - 2)^2} = \sqrt{125}$$

$$BC = \sqrt{(12 - 4)^2 + (4 - 8)^2} = \sqrt{80}$$

$$AC = \sqrt{(4 - 1)^2 + (8 - 2)^2} = \sqrt{45}$$

Clearly $AC^2 + BC^2 = AB^2$, so by the converse of the Pythagorean theorem there is a right angle at the vertex C . The largest angle of the triangle is 90° .

c) By the angle bisector theorem, $\frac{AP}{BP} = \frac{AC}{BC} = \frac{\sqrt{45}}{\sqrt{80}} = \frac{3}{4}$. Let $AP = 3x$, $BP = 4x$; then $3x + 4x = \sqrt{125}$, so $x = \frac{\sqrt{125}}{7}$ and $AP = \frac{15\sqrt{5}}{7}$.

By the theorem on the exterior angle bisector, $\frac{AQ}{BQ} = \frac{AC}{BC} = \frac{3}{4}$, so Q lies beyond A . Let $AQ = 3y$, $BQ = 4y$; then $4y - 3y = \sqrt{125}$, so $y = \sqrt{125}$ and $AQ = 15\sqrt{5}$. So

$$PQ = AP + AQ = \frac{15\sqrt{5}}{7} + 15\sqrt{5} = \frac{120\sqrt{5}}{7}$$

Since the interior and exterior angle bisectors are always perpendicular to each other, PQ is the hypotenuse of the right triangle PCQ , so $CP^2 + CQ^2 = PQ^2$:

$$CP^2 + CQ^2 = \left(\frac{120\sqrt{5}}{7}\right)^2 = \frac{120^2 \cdot 5}{49} = \frac{72\,000}{49} \approx 1469.39$$

9. Find the values of the following definite integrals.

$$\text{a) } \int_{-1}^4 (x^2 - 2x + 2) dx \quad \text{b) } \int_{-\pi}^{\pi} (x + \cos x) dx \quad \text{c) } \int_1^2 \left(e^x - \frac{1}{x}\right) dx \quad \text{d) } \int_0^1 (2x + 1)^4 dx$$

(16 points)

Solution: a)

$$\int_{-1}^4 (x^2 - 2x + 2) dx = \left[\frac{x^3}{3} - x^2 + 2x\right]_{-1}^4 = \left(\frac{64}{3} - 16 + 8\right) - \left(-\frac{1}{3} - 1 - 2\right) = \frac{50}{3}$$

b)

$$\int_{-\pi}^{\pi} (x + \cos x) dx = \left[\frac{x^2}{2} + \sin x\right]_{-\pi}^{\pi} = \left(\frac{\pi^2}{2} + \sin \pi\right) - \left(\frac{\pi^2}{2} + \sin(-\pi)\right) = 0$$

c)

$$\int_1^2 \left(e^x - \frac{1}{x}\right) dx = [e^x - \ln|x|]_1^2 = (e^2 - \ln 2) - (e - \ln 1) = e^2 - e - \ln 2$$

d)

$$\begin{aligned} \int_0^1 (2x + 1)^4 dx &= \int_0^1 (16x^4 + 32x^3 + 24x^2 + 8x + 1) dx = \left[\frac{16x^5}{5} + 8x^4 + 8x^3 + 4x^2 + x\right]_0^1 \\ &= \frac{16}{5} + 8 + 8 + 4 + 1 = \frac{121}{5} \end{aligned}$$