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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Of two convex polygons, the first one has 540 more diagonals than the second one, and the first one has three times as many vertices as the second one. How many vertices does each polygon have?

(11 points)

Solution: Let the second polygon have n vertices; then the first one has $3n$ vertices. A convex polygon with k vertices has $\frac{k(k-3)}{2}$ diagonals, so

$$\begin{aligned} \frac{3n\,(3n-3)}{2} - \frac{n\,(n-3)}{2} &= 540 \\ 9n^2 - 9n - n^2 + 3n &= 1080 \\ 4n^2 - 3n - 540 &= 0 \end{aligned}$$

By the quadratic formula,

$$n_{1,2} = \frac{3 \pm \sqrt{9 - 4 \cdot 4 \cdot (-540)}}{8} = \frac{3 \pm 93}{8}$$

Only a positive integer can be a solution, so $n = 12$. The first polygon has 36 vertices and the second one has 12. (Check: $\frac{36 \cdot 33}{2} - \frac{12 \cdot 9}{2} = 594 - 54 = 540$.)

2. The following notice can be read on the door of the snack bar at the Miskolc railway station:

Open 3:30 – 23:30. Closed between 7:30 – 8:30 and 19:30 – 20:30 because of the change of shifts!

a) What is the probability that we find the snack bar open, if we arrive at its door at a random time between 7 in the morning and 9 in the evening?

b) We bought a pastry for 200 forints. We paid with the exact amount. In how many ways could we do this, if we did not use any coin smaller than 20 forints, and the order does not matter? (Hungarian coins are worth 5, 10, 20, 50, 100, and 200 forints.)

c) The next customer wants to buy three pastries. The choices are: walnut crescent, jam-filled pastry, poppy seed roll, sour cherry strudel, and cocoa roll. What is the probability that we guess correctly what he will buy, if we assume that each of his possible choices is equally likely?

(12 points)

Solution: a) We represent the time between 7 and 21 o'clock with the interval $[7, 21]$ on the number line; its length is 14. During this time the snack bar is closed for 2 hours altogether, so the favorable moments form three segments of total length 12 (gray in the figure).



The probability is $p = \frac{12}{14} = \frac{6}{7}$.

b) We can use coins worth 200, 100, 50, and 20 forints. We list the cases according to the largest coin used.

Largest coin 200: 200.

Largest coin 100: $100 + 100$, $100 + 50 + 50$, $100 + 20 + 20 + 20 + 20 + 20$.

Largest coin 50: $50 + 50 + 50 + 50$, $50 + 50 + 20 + 20 + 20 + 20 + 20$.

Largest coin 20: $20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20$.

So we could pay the 200 forints in 7 ways.

c) The next customer chooses from 5 kinds of pastries. The problem does not say otherwise, so he may buy more than one of the same kind. If he buys three of the same kind, that is 5 cases. If he buys two of one kind and one of another, that is $5 \cdot 4 = 20$ cases: 5 ways to choose the kind he buys two of, and then 4 ways to choose the third pastry. If he buys three different pastries, that is $\binom{5}{3} = 10$ cases. Altogether there are $5 + 20 + 10 = 35$ cases, so the probability of guessing correctly is $\frac{1}{35}$.

Remark: With the formula for the number of combinations with repetition, $\binom{n+k-1}{k}$, we get the number of all cases immediately: $\binom{5+3-1}{3} = \binom{7}{3} = 35$.

3. Four points are given in the coordinate plane: $A_1\,(0,5)$, $B_1\,(1,7)$, $A_2\,(3,4)$, $B_2\,(5,5)$. We know that A_1A_2 and B_1B_2 are chords of two concentric circles.

a) Where is the common center of the circles?

b) We reflect the polygonal path $A_1B_1A_2B_2$ across the bisector of the first quadrant (the line $y = x$), and then we reflect both the original and the reflected figure across both axes and across the origin. What are the perimeter and the area of the polygon we get this way?

(14 points)

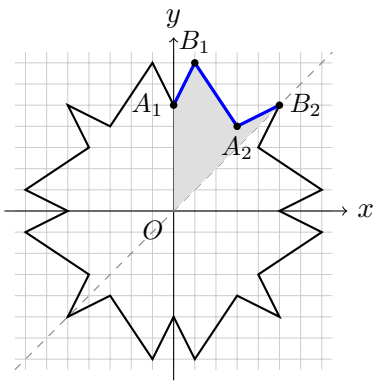
Solution: a) The common center lies on the perpendicular bisector f_1 of the chord A_1A_2 and also on the perpendicular bisector f_2 of the chord B_1B_2. So we need the intersection of these two lines.

The midpoint of A_1A_2 is $F_1\left(\frac{3}{2}, \frac{9}{2}\right)$, and $\overrightarrow{A_1A_2} = (3, -1)$ is a normal vector of f_1. So the equation of f_1 is $3x - y = 3 \cdot \frac{3}{2} - \frac{9}{2}$, that is, $3x - y = 0$.

The midpoint of B_1B_2 is $F_2\,(3,6)$, and $\overrightarrow{B_1B_2} = (4, -2)$ is a normal vector of f_2. So the equation of f_2 is $4x - 2y = 12 - 12$, that is, $4x - 2y = 0$.

Both lines pass through the origin (and they are not the same line), so the origin is the common center of the circles. (Indeed, $OA_1 = OA_2 = 5$ and $OB_1 = OB_2 = \sqrt{50}$.)

b) The figure shows the polygon we get.



The point B_2 lies on the line $y = x$, and A_1 lies on the y-axis. The reflection across $y = x$ doubles the area of the pentagon $OA_1B_1A_2B_2$ (shaded), and after all the reflections the polygon consists of 8 copies of this pentagon. We compute the area t of the pentagon by the shoelace formula (or by subtracting right triangles and a trapezoid from a 5×7 rectangle):

$$t = \frac{1}{2}\left|(0 \cdot 5 - 5 \cdot 0) + (5 \cdot 4 - 3 \cdot 5) + (3 \cdot 7 - 1 \cdot 4) + (1 \cdot 5 - 0 \cdot 7) + (0 \cdot 0 - 0 \cdot 5)\right| = \frac{1}{2}(5 + 17 + 5) = 13.5$$

So the area of the polygon is $T = 8t = 8 \cdot 13.5 = 108$.

Because of the reflections, the perimeter of the polygon is 8 times the length of the path $A_1B_1A_2B_2$:

$$K = 8\,(A_1B_1 + B_1A_2 + A_2B_2) = 8\left(\sqrt{5} + \sqrt{13} + \sqrt{5}\right) = 16\sqrt{5} + 8\sqrt{13} \approx 64.6$$

4. In the sequence 1, 6, 13, 24, 37, 54, 73, 96, 121, ..., each term after the second one is obtained by adding six times the index of the term between them to the term two places before it. We already know that the terms in even positions can be written as a quadratic function of their index. What is the 100th term of the sequence?

(14 points)

Solution: By the description, $a_n = a_{n-2} + 6\,(n-1)$. (For example, $a_3 = a_1 + 6 \cdot 2 = 13$ and $a_4 = a_2 + 6 \cdot 3 = 24$.) For even n, we look for the terms in the form $a_n = an^2 + bn + c$ with real coefficients $a \neq 0$, b, c. Since $a_2 = 6$, $a_4 = 24$, and $a_6 = 54$,

$$\begin{aligned} 4a + 2b + c &= 6 \\ 16a + 4b + c &= 24 \\ 36a + 6b + c &= 54 \end{aligned}$$

Subtracting each equation from the next one and dividing by 2, we get $6a + b = 9$ and $10a + b = 15$. Subtracting again, $4a = 6$, so $a = \frac{3}{2}$, and then $b = 0$ and $c = 0$. So for even n, $a_n = \frac{3}{2}n^2$ (check: $a_8 = 96$), and

$$a_{100} = \frac{3}{2} \cdot 100^2 = 15\,000$$

Part II

5. Three lines are given by their equations:

$$a:\ x + 3y = 7, \qquad b:\ 3x - y = -7, \qquad c:\ 3x + 4y = 8$$

a) Show that the three lines determine a right triangle.

b) How far is the vertex of the right angle from the midpoint of the opposite side?

c) What is the radius of the inscribed circle of the triangle?

(16 points)

Solution: a) Normal vectors of the lines are $\mathbf{n}_a = (1, 3)$, $\mathbf{n}_b = (3, -1)$, $\mathbf{n}_c = (3, 4)$. Since $\mathbf{n}_a \cdot \mathbf{n}_b = 1 \cdot 3 + 3 \cdot (-1) = 0$, these two normal vectors are perpendicular, so the lines a and b are perpendicular too.

We still have to check that the three lines really determine a triangle. $\mathbf{n}_c$ is not a multiple of $\mathbf{n}_a$ or of $\mathbf{n}_b$, so c is not parallel to a or to b. The solution of the system $x + 3y = 7$, $3x - y = -7$ is $x = -\frac{7}{5}$, $y = \frac{14}{5}$, so the lines a and b meet at $C\left(-\frac{7}{5}, \frac{14}{5}\right)$. C does not lie on c, because $3 \cdot \left(-\frac{7}{5}\right) + 4 \cdot \frac{14}{5} = 7 \neq 8$. So the three lines do not pass through one point, and they determine a right triangle.

Remark: The perpendicularity of a and b is necessary, but not sufficient.

b) From the system $3x - y = -7$, $3x + 4y = 8$ we get the vertex $A\left(-\frac{4}{3}, 3\right)$, and from the system $x + 3y = 7$, $3x + 4y = 8$ we get the vertex $B\left(-\frac{4}{5}, \frac{13}{5}\right)$. The right angle is at C. The midpoint of AB is

$$F\left(\frac{-\frac{4}{3} - \frac{4}{5}}{2}, \frac{3 + \frac{13}{5}}{2}\right) = F\left(-\frac{16}{15}, \frac{14}{5}\right)$$

so the distance in question is

$$CF = \sqrt{\left(-\frac{16}{15} + \frac{7}{5}\right)^2 + \left(\frac{14}{5} - \frac{14}{5}\right)^2} = \frac{5}{15} = \frac{1}{3}$$

Remark: The midpoint of the hypotenuse is the center of the circumscribed circle, so $CF = \frac{AB}{2}$; indeed $AB = \frac{2}{3}$ (see part c).

c) The sides of the triangle are

$$AC = \sqrt{\left(-\frac{7}{5} + \frac{4}{3}\right)^2 + \left(\frac{14}{5} - 3\right)^2} = \frac{\sqrt{10}}{15}, \quad BC = \sqrt{\left(-\frac{7}{5} + \frac{4}{5}\right)^2 + \left(\frac{14}{5} - \frac{13}{5}\right)^2} = \frac{\sqrt{10}}{5}$$

$$AB = \sqrt{\left(-\frac{4}{5} + \frac{4}{3}\right)^2 + \left(\frac{13}{5} - 3\right)^2} = \frac{2}{3}$$

The area of the right triangle is half the product of its legs:

$$t = \frac{1}{2} \cdot \frac{\sqrt{10}}{15} \cdot \frac{\sqrt{10}}{5} = \frac{1}{15}$$

Half of the perimeter is

$$s = \frac{1}{2}\left(\frac{\sqrt{10}}{15} + \frac{\sqrt{10}}{5} + \frac{2}{3}\right) = \frac{2\sqrt{10} + 5}{15}$$

The area is $t = \varrho s$, where ϱ is the radius of the inscribed circle. So

$$\varrho = \frac{t}{s} = \frac{1}{2\sqrt{10} + 5} = \frac{2\sqrt{10} - 5}{15} \approx 0.088$$

6. A function is given by the formula

$$f(x) = \frac{\frac{x+2}{x-2} - \frac{x-2}{x+2}}{\frac{x-2}{x+2} + 1}$$

a) Find the largest set that can be the domain of f.

b) Find the range of f.

c) How many lattice points (points with integer coordinates) lie on the graph of f? List these points.

(16 points)

Solution: a) None of the denominators can be zero, so $x \neq 2$, $x \neq -2$, and $\frac{x-2}{x+2} + 1 \neq 0$. For the last one,

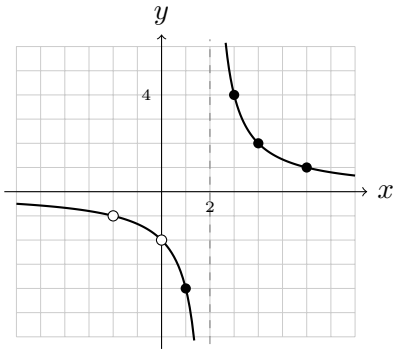
$$\frac{x-2}{x+2} + 1 = \frac{x - 2 + x + 2}{x+2} = \frac{2x}{x+2} \neq 0, \quad \text{that is,} \quad x \neq 0$$

So the largest possible domain is $\mathbb{R} \setminus \{-2, 0, 2\}$.

b) We simplify the formula:

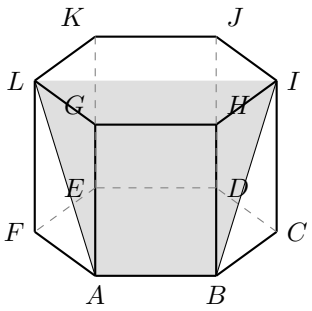
$$f(x) = \frac{(x+2)^2 - (x-2)^2}{(x-2)(x+2)} : \frac{2x}{x+2} = \frac{8x}{(x-2)(x+2)} \cdot \frac{x+2}{2x} = \frac{4}{x-2}$$

The function $g(x) = \frac{4}{x-2}$ is a transformation of $h(x) = \frac{1}{x}$, so its range is $\mathbb{R} \setminus \{0\}$, and it takes every value in its range exactly once. f agrees with g except at $x = -2$ and $x = 0$, where f is not defined. Since $g(-2) = -1$ and $g(0) = -2$, the range of f is $\mathbb{R} \setminus \{-2, -1, 0\}$.



c) We need the integers x in the domain for which $\frac{4}{x-2}$ is an integer. So $x - 2$ must be a divisor of 4: $x - 2 \in \{-4, -2, -1, 1, 2, 4\}$, that is, $x \in \{-2, 0, 1, 3, 4, 6\}$. The numbers -2 and 0 are not in the domain. For the other four, the values of $\frac{4}{x-2}$ are -4, 4, 2, 1. So four lattice points lie on the graph of f: $(1, -4)$, $(3, 4)$, $(4, 2)$, $(6, 1)$.

7. László is designing a box for a game called "Exact Step"; the box is a right prism whose bases are regular hexagons. The bottom face of the box is the regular hexagon $ABCDEF$, and the top face is the regular hexagon $GHIJKL$ (with G above A, H above B, and so on).



a) What would be the area of the side face $ABHG$, if the area of the cross-section $ABIL$ is 72 cm^2 and the distance between the parallel lines AB and IL is 8 cm?

b) What would be the angle between the base and the plane $ABIL$, if $AL = \sqrt{2} \cdot AB$?

(16 points)

Solution: *Note: In KoMaL, part a) gives only the area of the cross-section; the distance 8 cm between the lines AB and IL appears only in the solution. Without it, the area of $ABHG$ is not determined, so we added it to the problem.*

a) Let $AB = x$. Then $LI = 2x$, because it is a longest diagonal of the regular hexagon $GHIJKL$. The cross-section $ABIL$ is an isosceles trapezoid with parallel sides $AB = x$ and $LI = 2x$ and height 8 cm, so its area is

$$\frac{(x + 2x) \cdot 8}{2} = 12x = 72, \qquad \text{so} \qquad x = 6 \text{ cm}$$

Let M be the foot of the perpendicular from A to the line LI. M lies above the line FC of the bottom face, and the distance between the parallel lines AB and FC in the regular hexagon is $\frac{\sqrt{3}}{2}x$ (the height of an equilateral triangle of side x). So in the right triangle formed by A, M, and the point of the bottom face below M, the hypotenuse is $AM = 8$ and the horizontal leg is $\frac{\sqrt{3}}{2}x = 3\sqrt{3}$. The vertical leg is the height of the prism:

$$AG = \sqrt{8^2 - \left(3\sqrt{3}\right)^2} = \sqrt{64 - 27} = \sqrt{37}$$

So the area of the side face $ABHG$ is

$$t_{ABHG} = AB \cdot AG = 6\sqrt{37} \approx 36.5 \text{ cm}^2$$

b) Let the side of the hexagons be x again. Now $AL = \sqrt{2}\,x$. In the right triangle AFL we have $AF = x$, so $FL = \sqrt{2x^2 - x^2} = x$: the height of the prism is x.

The plane $ABIL$ meets the base along the line AB. The angle between the two planes is the angle between two lines perpendicular to AB, one in each plane. In the plane $ABIL$, take the height AM of the trapezoid; its projection to the base is perpendicular to AB, and its length is the distance between AB and FC, which is $\frac{\sqrt{3}}{2}x$. The vertical leg is the height x. So for the angle φ between the base and the plane $ABIL$,

$$\tan\varphi = \frac{x}{\frac{\sqrt{3}}{2}x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}, \quad \text{so} \quad \varphi \approx 49.1^\circ$$

8. a) The tangent lines to the graph of the function $f(x) = 4 - x^2$ at its points with x-coordinates 1 and -1, together with the x-axis, determine a triangle. We cut the triangle along the graph of the function. What is the total area of the regions on the right, on the left, and on the top of the curve?

   b) The cross-section of a tunnel can be well approximated by the region between the graph of $f(x) = 4 - x^2$ and the x-axis, where the unit of the coordinate system is 1 meter. What is the largest possible volume of a 6 meter long rectangular box that can be pushed through the tunnel, sliding on one of its faces?

(16 points)

Solution: a) $f(-1) = f(1) = 3$, so the tangent lines touch the graph at $A(-1, 3)$ and $B(1, 3)$. The slope of a tangent line is given by the derivative, $f'(x) = -2x$. Since $f'(-1) = 2$, the tangent line at A is $y - 3 = 2(x + 1)$, that is, $y = 2x + 5$. Since $f'(1) = -2$, the tangent line at B is $y - 3 = -2(x - 1)$, that is, $y = -2x + 5$. The two tangent lines meet at $P(0, 5)$, and they meet the x-axis at $Q\left(-\frac{5}{2}, 0\right)$ and $R\left(\frac{5}{2}, 0\right)$. So the area of the isosceles triangle PQR is

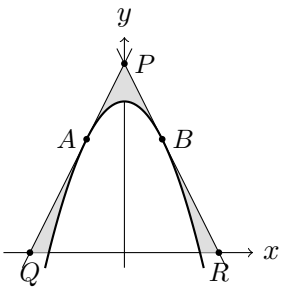
$$t_{PQR} = \frac{QR \cdot OP}{2} = \frac{5 \cdot 5}{2} = \frac{25}{2}$$

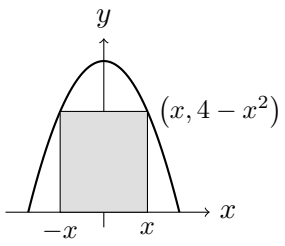
The graph of f meets the x-axis at $(-2, 0)$ and $(2, 0)$, so the area under the curve is

$$t_1 = \int_{-2}^{2} \left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$$

The total area of the regions in question (shaded in the left figure) is the difference:

$$t = \frac{25}{2} - \frac{32}{3} = \frac{11}{6}$$





b) The 6 meter long edges of the box must be horizontal, along the tunnel, so we need the rectangle of largest area that fits in the cross-section of the tunnel (right figure). If its upper right vertex is $\left(x, 4-x^2\right)$, its area is

$$t(x) = 2x\left(4-x^2\right) = 8x - 2x^3, \qquad 0 < x < 2$$

$t'(x) = 8 - 6x^2 = 0$ gives $x = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ in this interval. Here the derivative changes from positive to negative, so t has its maximum there. So the edges of the box of largest volume are $2x = \frac{4\sqrt{3}}{3}$, $4 - x^2 = \frac{8}{3}$, and 6 meters, and its volume is

$$V = \frac{4\sqrt{3}}{3} \cdot \frac{8}{3} \cdot 6 = \frac{64\sqrt{3}}{3} \approx 36.95 \text{ m}^3$$

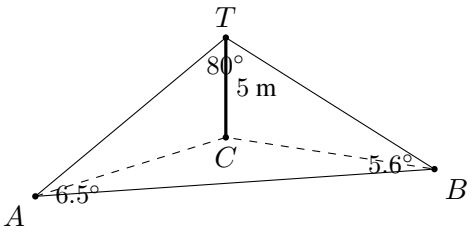
9. A 5 meter tall vertical pole stands on level ground. Its top is seen from the point A of the ground at an angle of elevation of $6.5°$, and from the point B at an angle of elevation of $5.6°$. From the top of the pole, the segment AB is seen at an angle of $80°$.

a) Describe briefly how we would find the area of the triangle ABC (C is the foot of the pole), if we knew the length of AB.

b) How long is the distance AB?

(16 points)

Solution: Let T be the top of the pole.



a) In the right triangle ACT we know $CT = 5$ m and the angle at A, so we can find AC with the tangent function: $AC = \frac{5}{\tan 6.5°}$. Similarly, in the right triangle BCT, $BC = \frac{5}{\tan 5.6°}$. If we also knew AB, then we would know all three sides of triangle ABC. By the law of cosines we could find the angle γ at C, and then

$$t_{ABC} = \frac{CA \cdot CB \cdot \sin\gamma}{2}$$

Remark: Knowing the three sides, we could also use Heron's formula.

b) In the right triangle ACT, $\sin 6.5° = \frac{5}{AT}$, so $AT = \frac{5}{\sin 6.5°} \approx 44.17$ m. In the right triangle BCT, $\sin 5.6° = \frac{5}{BT}$, so $BT = \frac{5}{\sin 5.6°} \approx 51.24$ m. By the law of cosines in triangle ABT,

$$AB^2 = \left(\frac{5}{\sin 6.5°}\right)^2 + \left(\frac{5}{\sin 5.6°}\right)^2 - 2 \cdot \frac{5}{\sin 6.5°} \cdot \frac{5}{\sin 5.6°} \cdot \cos 80°, \qquad \text{so} \qquad AB \approx 61.6 \text{ m}$$

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