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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. In one month, Péter made 18 calls and talked for 60 minutes within his phone network. Outside the network, he made 11 calls and talked for 45 minutes. He sent 10 text messages. Compare the two calling plans below ("Haver 1" and "Haver 2"; *haver* means *buddy*), and show by computation which one would have been cheaper for him in that month.

Name of the plan	"Haver 1"	"Haver 2"
Monthly fee, usable for calls	1710 Ft 100% within the network	3850 Ft 60% within the network 40% outside the network
Rate per minute until the monthly fee is used up	38 Ft	35 Ft
Rate per minute after the monthly fee is used up	40 Ft	38 Ft
Text message (SMS)	40 Ft	38 Ft
Connection fee per call	3 Ft	3 Ft

The monthly fee can not be used for the connection fees; those are always billed on top of the monthly fee.

(11 points)

Solution: "Haver 1": the monthly fee covers $1710 : 38 = 45$ minutes within the network. Péter talked $60 - 45 = 15$ minutes more within the network, which costs $15 \cdot 40 = 600$ Ft above the monthly fee. The monthly fee can not be used outside the network, so the 45 minutes outside the network cost $45 \cdot 40 = 1800$ Ft. The text messages cost $10 \cdot 40 = 400$ Ft, and the connection fees are $(18 + 11) \cdot 3 = 87$ Ft. The total cost is

$$1710 + 600 + 1800 + 400 + 87 = 4597 \text{ Ft}$$

"Haver 2": 60% of the monthly fee is $3850 \cdot 0.6 = 2310$ Ft, which covers $2310 : 35 = 66$ minutes within the network; Péter used only 60 of these. 40% of the monthly fee is $3850 \cdot 0.4 = 1540$ Ft, which covers $1540 : 35 = 44$ minutes outside the network; Péter talked one more minute, which costs 38 Ft. The text messages cost $10 \cdot 38 = 380$ Ft, and the connection fees are $29 \cdot 3 = 87$ Ft. The total cost is

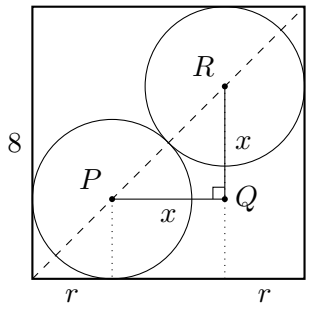
$$3850 + 38 + 380 + 87 = 4355 \text{ Ft}$$

So "Haver 2" would have been cheaper for him, by $4597 - 4355 = 242$ Ft.

2. A box has the shape of a prism with a square base, and the edge of its base is 8 cm. Two coins of equal diameter fit exactly on the square bottom of the box, lying flat, so that each coin touches two neighboring side faces of the box, and the two coins touch each other. What is the diameter of the coins?

(12 points)

Solution: Let the radius of the coins be r, and let P and R be the centers of the two coins, as in the figure. Each coin touches two neighboring sides, so its center is at distance r from both of these sides; the two centers lie on a diagonal of the square. The coins touch each other, so $PR = 2r$. Let Q be the point for which PQR is a right isosceles triangle with legs $PQ = QR = x$ parallel to the sides of the square.



By the Pythagorean theorem in triangle PQR,

$$(2r)^2 = x^2 + x^2, \quad \text{so} \quad x = \sqrt{2}\, r$$

The side of the square is 8 cm, so $r + x + r = 8$, that is, $2r + \sqrt{2}r = 8$. Therefore

$$r = \frac{8}{2+\sqrt{2}} = 8 - 4\sqrt{2}, \quad \text{and the diameter is} \quad d = 2r = 16 - 8\sqrt{2} \approx 4.7 \text{ cm}$$

3. Solve the following equations over the set of real numbers.

a) $\cos 2x + 5 \sin x + 2 = 0$ *(7 points)*

b) $2 \log_5 x - \dfrac{3}{\log_5 x} = 5$ *(7 points)*

Solution: a) We use $\cos 2x = \cos^2 x - \sin^2 x$ and $\cos^2 x = 1 - \sin^2 x$, so $\cos 2x = 1 - 2\sin^2 x$. The equation becomes

$$1 - 2\sin^2 x + 5 \sin x + 2 = 0, \quad \text{that is,} \quad 2\sin^2 x - 5 \sin x - 3 = 0$$

This is a quadratic equation in $\sin x$, with roots $\sin x = -\frac{1}{2}$ and $\sin x = 3$. The second one gives no solution, because $-1 \le \sin x \le 1$. From $\sin x = -\frac{1}{2}$,

$$x_1 = -\frac{\pi}{6} + 2k\pi, \quad x_2 = \frac{7\pi}{6} + 2l\pi, \quad \text{where } k, l \in \mathbb{Z}$$

These are indeed solutions of the original equation.

b) The equation is defined for $x > 0$, $x \neq 1$. We multiply both sides by $\log_5 x$ and rearrange:

$$2 (\log_5 x)^2 - 5 \log_5 x - 3 = 0$$

This is the same quadratic equation as in part a), now in $\log_5 x$. So $\log_5 x = -\frac{1}{2}$, which gives $x_1 = 5^{-1/2} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$, or $\log_5 x = 3$, which gives $x_2 = 125$. Both satisfy the original equation.

4. Thirty students wrote a 10-point test. The results were the following:

score	3	4	5	6	7	8	9	10
frequency	2	3	7	2	2	5	5	4

a) The teacher congratulated the class, because $\frac{3}{5}$ of the students scored more than one of the averages (mean, median, or mode). Which one did the teacher mean? *(6 points)*

b) Two students wrote the test later. With their scores included, the median increased by 0.5 and the mean increased by 0.075. What could their scores be? *(8 points)*

Solution: a) The mean of the scores is

$$\frac{3\cdot 2+4\cdot 3+5\cdot 7+6\cdot 2+7\cdot 2+8\cdot 5+9\cdot 5+10\cdot 4}{30}=\frac{204}{30}=6.8$$

16 students scored more than this. Since $\frac{16}{30}<\frac{3}{5}$, the teacher did not mean the mean.

The median is the average of the 15th and 16th scores (in increasing order): $\frac{7+7}{2}=7$. 14 students scored more than this. Since $\frac{14}{30}<\frac{3}{5}$, the teacher did not mean the median.

The mode (the most frequent score) is 5. 18 students scored more than this, and $\frac{18}{30}=\frac{3}{5}$. So the teacher meant the mode.

b) Let the two scores be x and y. The new mean is $6.8+0.075=6.875$, so

$$\frac{204+x+y}{32}=6.875,\qquad \text{so}\qquad x+y=16$$

The scores are at most 10, so the possibilities are $16=6+10=7+9=8+8$. The new median is the average of the 16th and 17th scores. For 6 and 10, or for 7 and 9, the 16th and 17th scores are both 7, so the median does not change. For 8 and 8, the 16th score is 7 and the 17th is 8, so the median is 7.5; it increased by 0.5. So both students scored 8 points.

Part II

5. In a card game, the cards are numbered from 1 to 104, and each card carries a number of penalty points according to the following rules:
   - card number 55 carries 7 penalty points;
   - the other cards whose number ends in 5 carry 2 penalty points each;
   - the other cards with a two-digit number made of two equal digits carry 5 penalty points each;
   - the cards whose number ends in 0 carry 3 penalty points each;
   - all other cards carry 1 penalty point each.

   At the beginning of the game, 4 cards are turned face up, every player gets 10 cards, and the remaining cards go to the stock.

   a) How many penalty points are on the cards altogether? *(4 points)*

   b) What is the probability that each of the four cards turned face up carries more than one penalty point? *(6 points)*

   c) With six players, what is the probability that there is at least one card in the stock that carries 5 or more penalty points? *(6 points)*

   Solution: a) Card 55 carries 7 points (1 card). The cards 5, 15, 25, 35, 45, 65, 75, 85, 95 carry 2 points each (9 cards). The cards 11, 22, 33, 44, 66, 77, 88, 99 carry 5 points each (8 cards). The cards 10, 20, 30, ..., 100 carry 3 points each (10 cards). That is 28 cards; the other 76 cards carry 1 point each. So the total number of penalty points is

$$1\cdot 7+9\cdot 2+8\cdot 5+10\cdot 3+76\cdot 1=171$$

b) 28 cards carry more than one penalty point. The number of favorable cases is $\binom{28}{4}$, and the number of all cases is $\binom{104}{4}$, so

$$p = \frac{\binom{28}{4}}{\binom{104}{4}} \approx 0.00445$$

c) $8+1=9$ cards carry 5 or more penalty points, and $104-9=95$ cards carry fewer. With six players, $4+6\cdot 10=64$ cards are used, so 40 cards are in the stock. It is easier to compute the probability of the complementary event $\overline{A}$: there is no card with 5 or more penalty points in the stock, that is, all 40 cards of the stock come from the 95 other cards.

$$p\left(\overline{A}\right) = \frac{\binom{95}{40}}{\binom{104}{40}} \approx 0.01$$

So the probability that there is at least one card with 5 or more penalty points in the stock is $p(A) = 1 - p\left(\overline{A}\right) \approx 0.99$.

6. A factory makes one-liter measuring cups in the shape of a cylinder. What should the radius of the base and the height of the cylinder be, so that the amount of material used is minimal,

   a) for a cup without a lid; *(8 points)*

   b) for a cup with a lid? *(8 points)*

   Give the answers to the nearest millimeter.

   Solution: Let the radius of the base be r and the height be m, in decimeters. The volume is $V = r^2\pi m = 1$ dm^3, so $m = \frac{1}{r^2\pi}$. The amount of material is minimal when the surface area is minimal.

   a) Without a lid, the surface area is

$$A(r) = r^2\pi + 2r\pi m = r^2\pi + \frac{2}{r}, \qquad r > 0$$

$$A'(r) = 2r\pi - \frac{2}{r^2}, \qquad A''(r) = 2\pi + \frac{4}{r^3} > 0$$

$A'(r) = 0$ gives $r^3 = \frac{1}{\pi}$, that is, $r = \sqrt[3]{\frac{1}{\pi}}$, and there $A'' > 0$, so this is a minimum. Then

$$m = \frac{1}{r^2\pi} = \frac{r^3}{r^2} = r \approx 0.683 \text{ dm}$$

So the material used is minimal when $r = m \approx 68$ mm.

   b) With a lid, the surface area is

$$A(r) = 2r^2\pi + 2r\pi m = 2r^2\pi + \frac{2}{r}, \qquad A'(r) = 4r\pi - \frac{2}{r^2}, \qquad A''(r) = 4\pi + \frac{4}{r^3} > 0$$

$A'(r) = 0$ gives $r^3 = \frac{1}{2\pi}$, that is, $r = \sqrt[3]{\frac{1}{2\pi}} \approx 0.542$ dm, and this is a minimum. Then

$$m = \frac{1}{r^2\pi} = \frac{2r^3}{r^2} = 2r = \sqrt[3]{\frac{4}{\pi}} \approx 1.084 \text{ dm}$$

So the material used is minimal when $r \approx 54$ mm and $m \approx 108$ mm. (The height equals the diameter.)

7. The parabola with equation $y = ax^2 + bx + c$ passes through the point $(8, 0)$ and through the origin. The slope of its tangent line at the origin is 2.

   a) Find a, b, and c. *(6 points)*

   b) We draw a secant line through the origin and the point of the parabola with x-coordinate 6. What is the area of the region enclosed by this secant line and the parabola? *(10 points)*

   Solution: a) The parabola passes through $(0, 0)$ and $(8, 0)$:

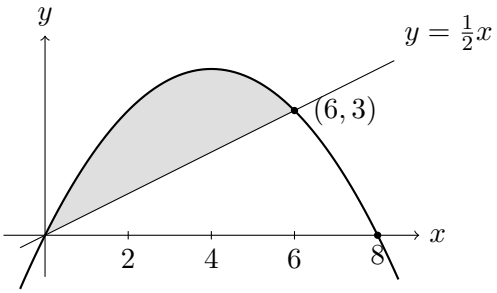
$$0 = a \cdot 0 + b \cdot 0 + c, \qquad 0 = 64a + 8b + c$$

   So $c = 0$ and $a = -\frac{1}{8}b$. The slope of the tangent line at the origin is the value of the derivative at 0: $f'(x) = 2ax + b$, so $f'(0) = b = 2$. Therefore

$$a = -\frac{1}{4}, \qquad b = 2, \qquad c = 0$$

   b) $f(x) = -\frac{1}{4}x^2 + 2x$, so $f(6) = 3$. The secant line through the origin and $(6, 3)$ is $y = \frac{1}{2}x$. On $[0, 6]$ the parabola is above the line, so the area is

$$\int_0^6 \left(-\frac{1}{4}x^2 + 2x - \frac{1}{2}x\right) dx = \int_0^6 \left(-\frac{1}{4}x^2 + \frac{3}{2}x\right) dx = \left[-\frac{x^3}{12} + \frac{3x^2}{4}\right]_0^6 = -18 + 27 = 9$$



   The region enclosed by the parabola and the secant line has area 9 square units.

8. a) Prove that $64 \mid 9^n - 8n - 1$ for every natural number n. *(8 points)*

   b) Is it true that $2013^2 + 2014 \mid 2013^{2014} - 2013$? Justify your answer. *(8 points)*

   Solution: a) We use mathematical induction. For $n = 0$: $9^0 - 0 - 1 = 0$, which is divisible by 64. (Also, for $n = 1$ we get 0, and for $n = 2$ we get 64.)

   Suppose that the statement is true for $n = k$, that is, $64 \mid 9^k - 8k - 1$. We need to show that $64 \mid 9^{k+1} - 8(k + 1) - 1$. We rewrite this expression:

$$9^{k+1} - 8(k + 1) - 1 = 9 \cdot 9^k - 8k - 9 = 9\left(9^k - 8k - 1\right) + 64k$$

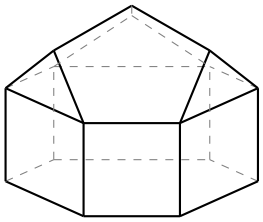
   The first term is divisible by 64 by the induction hypothesis, and the second term is clearly divisible by 64. So their sum is divisible by 64 too, which proves the statement.

   b) Yes, it is true. Let $a = 2013$. Then the left side is $a^2 + a + 1$, and the right side is

$$a^{2014} - a = a\left(a^{2013} - 1\right) = a\left(\left(a^3\right)^{671} - 1\right) = a\left(a^3 - 1\right)\left(a^{3 \cdot 670} + a^{3 \cdot 669} + \cdots + a^3 + 1\right)$$

   Since $a^3 - 1 = (a - 1)\left(a^2 + a + 1\right)$, the right side is divisible by $a^2 + a + 1$.

9. At the top vertex of a structure that serves as a shelter for hikers, three regular pentagons meet, as shown in the figure. The sides of the regular pentagons are 1.5 m long. The side faces of the structure are rectangles, whose sides perpendicular to the ground are also 1.5 m long.



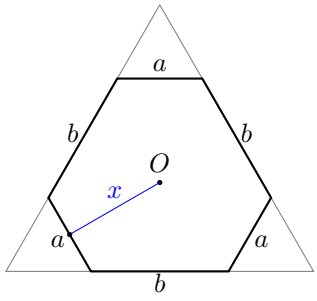
a) How tall is the structure? *(8 points)*

b) What is the area of the base of the structure? *(8 points)*

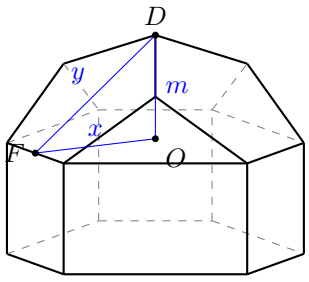
Solution: a) Let $a = 1.5$ m. The roof can be thought of as part of a regular dodecahedron (whose faces are regular pentagons), with three pentagons meeting at the top vertex D. Between two neighboring pentagons there is a triangle whose sides are a, a, and a diagonal b of the pentagon; its angle between the two sides a is an angle of the regular pentagon, $\frac{3\cdot 180^\circ}{5} = 108^\circ$. By the law of cosines,

$$b^2 = a^2 + a^2 - 2a^2 \cos 108^\circ, \qquad \text{so} \qquad b \approx 2.43 \text{ m}$$

So the base of the structure is a hexagon whose sides are a, b, a, b, a, b, in this order. The structure is symmetric under a rotation by 120° about its vertical axis through D, so the opposite sides of the hexagon are parallel, and all its angles are 120°. Therefore the hexagon can be completed to an equilateral triangle of side $b + 2a$ by adding three equilateral triangles of side a at its corners.



Let O be the center of the hexagon at the height of the top of the walls, let F be the midpoint of the top edge of a wall of width a, let $x = OF$, $y = FD$, and $m = OD$.



y is the distance between a vertex and the opposite side of a regular pentagon. In the pentagon, the side at F and the diagonal from its endpoint to D form an angle of $108^\circ - 36^\circ = 72^\circ$ (an angle of the pentagon minus a base angle of the isosceles triangle cut off by the diagonal), so

$$y = b \sin 72^\circ \approx 2.31 \text{ m}$$

x is the distance from the center of the big equilateral triangle to the cut-off line at a corner. This is two thirds of the median of the big triangle (the distance from the center to the corner) minus the height of a small triangle of side a:

$$x = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} (b + 2a) - \frac{\sqrt{3}}{2} a \approx 1.83 \text{ m}$$

In the right triangle DFO, by the Pythagorean theorem,

$$m = \sqrt{y^2 - x^2} \approx 1.40 \text{ m}$$

So the height of the structure is $a + m \approx 1.5 + 1.40 = 2.9$ m.

b) The area of the base is the area of the equilateral triangle of side $b + 2a$ minus three times the area of an equilateral triangle of side a:

$$T = \frac{\sqrt{3}}{4} (b + 2a)^2 - 3 \cdot \frac{\sqrt{3}}{4} a^2 \approx 9.83 \text{ m}^2$$

Note: KoMaL gives 9.84 *m*^{2}*, computed with the rounded value* $b \approx 2.43$*; with the exact value* $b = \frac{1+\sqrt{5}}{2} a$ *the area is* 9.83 *m*^{2}.

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