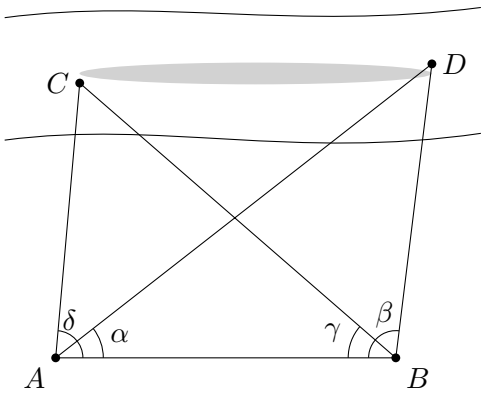
Advanced Level Practice Set - 2013 (September 2013)

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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. A surveyor had to give a reasonably good estimate of the length CD of the long, narrow island in the middle of the river shown in the figure. The river is fast and full of whirlpools, so the surveyor did not cross over to the island, but solved the problem another way. On the flat riverbank he marked the points A and B, 100 meters apart, as in the figure. Then he measured the angles $\alpha = 38°$, $\delta = 85°$, $\beta = 97°$, and $\gamma = 41°$, rounded to the nearest degree, and computed the approximate length of the island. Let us compute it too.



(12 points)

2. a) State the converse of the following statement, and decide whether the converse is true or false:

If a quadrilateral is a kite, then one of its diagonals bisects the other diagonal.

b) Negate the following sentence:

A dog that barks does not bite.

c) Anett, Béla, and Cecília are very good friends. Their professions are pianist, teacher, and engineer, but not necessarily in this order. One of them lives in Budapest, another one in Debrecen, and the third one in Szombathely. Next summer Béla and Cecília would like to travel to Szombathely to visit Anett, who lives there alone in her big house with a garden, and they would like to spend their summer vacation with her. The pianist promised to give Béla one of the pianist's newest CDs then. The engineer, who does not live in Budapest, likes to play chess with the pianist from Debrecen; hopefully there will be a chance for that during the vacation too. Who lives where, and what is each person's profession?

(12 points)

3. a) Which number has the property that if we raise it to the power of its base-10 logarithm, we get one hundred million times its square?

   b) We deposit one million forints in a bank at an annual interest rate of 2.5%. The interest is added to the principal on the last day of each year, after a 16% interest tax and a 6% health care contribution (which is also charged on the interest) have been deducted from it. At the end of which year will our deposit exceed 1.2 million forints?

   (13 points)

4. Consider the following function:

$$f(x) = \begin{cases} \dfrac{12}{x} & \text{if } x < -3 \\ \left|x^2 - 4\right| & \text{if } -3 \leq x < 3 \\ -\dfrac{12}{x} & \text{if } x \geq 3 \end{cases}$$

   a) Graph the function on the interval $[-12, 12]$.

   b) Find the zeros of the function.

   c) Find the range of the function.

   d) Find the absolute maximum and minimum points and values of the function, and also its local maximum and minimum points and values.

   e) Is f an even function?

   (14 points)

Part II

5. We have a tetrahedral die (a regular tetrahedron with uniform mass distribution, with the numbers 1, 2, 3, and 4 written on its faces) and an octahedral die (a regular octahedron with uniform mass distribution, with the natural numbers from 1 to 8 written on its faces). We roll the two dice together. The outcome of an experiment is the sum of the numbers written on the faces touching the table.

   a) Draw a bar chart of the distribution of the random variable defined this way, and compute its expected value.

   b) What is the probability that the number on the face of the octahedral die touching the table is greater than the number on the face of the tetrahedral die touching the table?

   c) If we repeat the experiment many times, what is the probability that the outcome is 9 for the first time in the eighth experiment?

   d) We did the experiment one hundred times. We recorded the outcomes and got the frequency table below. For which outcome is the absolute difference between the relative frequency and the theoretical probability, rounded to three decimal places, the largest?

outcome	2	3	4	5	6	7	8	9	10	11	12
frequency	0	6	10	13	15	8	16	14	11	5	2

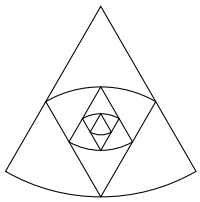
   (16 points)

6. a) The difference of the cubes of two positive integers is 169. What are these numbers?

   b) Prove that if we subtract a positive integer from its fifth power, the difference is always divisible by each of the three smallest positive primes.

   (16 points)

7. We draw new, similar circular sectors into a circular sector of radius 1000 meters and central angle $60°$, as shown in the figure, and continue this "forever".



a) What are the area and the perimeter of the 12th inscribed (that is, the 13th drawn) sector?

b) Consider the sequence of the perimeters of the sectors, whose first term is the perimeter of the sector of radius 1000 meters. Compute the sum of its first 13 terms.

c) Consider the sequence of the areas of the sectors, whose first term is the area of the sector of radius 1000 meters, and the further sectors are obtained as described above. Denote these areas by T_1, T_2, and so on. What is the limit of $S_n = T_1 + T_2 + \cdots + T_n$ as $n \to \infty$?

(16 points)

8. The vertices of a triangle are $A\,(2, 5)$, $B\,(7, -7)$, and $C\,(-4, -3)$. The interior angle bisector from vertex A intersects the side BC at the point D.

   a) Find the equation of the interior angle bisector through A.

   b) Find the coordinates and the length of the vector $\overrightarrow{AD}$.

   c) Find the angle between the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

   d) Can there be a triangle ABC in which the segment of the interior angle bisector from A inside the triangle is longer than the side AB? If yes, what is the condition for this, in terms of the angles of the triangle?

   (16 points)

9. In a class of 30 students, 11 boys and 8 girls play a musical instrument. 9 boys and 14 girls play sports competitively, and 13 boys attend the math club. Three of the girls do not play any sport at all, but they attend the math club.

   a) How many boys and how many girls are there in the class?

   b) Each of the 8 girls who play an instrument marked exactly three of the others as friends on a social networking site (and those marked her too). Illustrate such a network of friendships with a graph.

   c) Five of the boys who play sports competitively play water polo. When we asked them, they told us that they are also registered on the social networking site mentioned in part b). Each of them told us how many of the other four are mutual friends with him on the site. The answers were 4, 4, 3, 2, 1. What can we conclude?

   d) Another class of 30 students has only 13 boys. The class is preparing a dance performance for a show, in which 15 couples will dance. The necessary number of girls will dress as boys. The girls who dress as boys, and the couples, are chosen by drawing lots. In how many ways can the couples be formed?

   e) What is the probability that, as a result of the drawing in part d), Mária Kis, a student in the class, dances with a girl dressed as a boy?

   f) What is the probability that, as a result of the drawing in part d), at most two of the three competitive dancer girls, Mária Kis, Anita Nagy, and Karola Tóth, have to dress as boys?

   g) In how many ways can eight different gifts be given out by drawing lots among the students of the class in part d), if each student can get at most one gift? In how many of these cases is there at least one boy among the winners?

   (16 points)

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