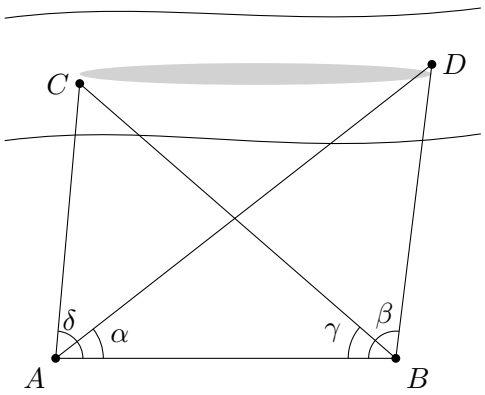
author: József Czinki
translated from: KoMaL, September 2013, pages 329-332
Pdf file: KoMaL, September 2013, pages 329-332

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. A surveyor had to give a reasonably good estimate of the length CD of the long, narrow island in the middle of the river shown in the figure. The river is fast and full of whirlpools, so the surveyor did not cross over to the island, but solved the problem another way. On the flat riverbank he marked the points A and B, 100 meters apart, as in the figure. Then he measured the angles $\alpha = 38°$, $\delta = 85°$, $\beta = 97°$, and $\gamma = 41°$, rounded to the nearest degree, and computed the approximate length of the island. Let us compute it too.



(12 points)

Solution: In triangle ABC, the angle at C is $180° - 41° - 85° = 54°$. By the law of sines,

$$\frac{AC}{\sin 41°} = \frac{100}{\sin 54°}, \quad \text{so} \quad AC = \frac{100 \sin 41°}{\sin 54°} \approx 81.09 \text{ m}$$

In triangle ABD, the angle at D is $180° - 38° - 97° = 45°$. By the law of sines,

$$\frac{AD}{\sin 97°} = \frac{100}{\sin 45°}, \quad \text{so} \quad AD = \frac{100 \sin 97°}{\sin 45°} \approx 140.37 \text{ m}$$

In triangle ACD, the angle at A is $\delta - \alpha = 47°$. By the law of cosines,

$$CD^2 = AC^2 + AD^2 - 2 \cdot AC \cdot AD \cdot \cos(\delta - \alpha), \quad \text{so} \quad CD \approx 104 \text{ m}$$

The island is about 104 meters long.

2. a) State the converse of the following statement, and decide whether the converse is true or false:

If a quadrilateral is a kite, then one of its diagonals bisects the other diagonal.

b) Negate the following sentence:

A dog that barks does not bite.

c) Anett, Béla, and Cecília are very good friends. Their professions are pianist, teacher, and engineer, but not necessarily in this order. One of them lives in Budapest, another one in Debrecen, and the third one in Szombathely. Next summer Béla and Cecília would like to travel to Szombathely to visit Anett, who lives there alone in her big house with a garden, and they would like to spend their summer vacation with her. The pianist promised to give Béla one of the pianist's newest CDs then. The engineer, who does not live in Budapest, likes to play chess with the pianist from Debrecen; hopefully there will be a chance for that during the vacation too. Who lives where, and what is each person's profession?

(12 points)

Solution: a) The converse: *If one diagonal of a quadrilateral bisects the other diagonal, then the quadrilateral is a kite.*

The converse is false. A counterexample: in a parallelogram that is not a rhombus, each diagonal bisects the other one, but the parallelogram is not a kite.

b) The negation: *There is at least one dog that barks and bites.*

Remark: The sentence "A dog that barks does not bite" says that *every* dog that barks does not bite (universal quantifier). Its negation says that *there exists* a dog that barks and bites (existential quantifier).

c) Anett has a house in Szombathely, so she lives in Szombathely. The pianist lives in Debrecen and will give Béla a CD, so Béla does not live in Debrecen; therefore he lives in Budapest. This means that the pianist from Debrecen is Cecília. The engineer does not live in Budapest, so the engineer is Anett, and then Béla is the teacher. In summary:

Anett	Szombathely	engineer
Béla	Budapest	teacher
Cecília	Debrecen	pianist

3. a) Which number has the property that if we raise it to the power of its base-10 logarithm, we get one hundred million times its square?

b) We deposit one million forints in a bank at an annual interest rate of 2.5%. The interest is added to the principal on the last day of each year, after a 16% interest tax and a 6% health care contribution (which is also charged on the interest) have been deducted from it. At the end of which year will our deposit exceed 1.2 million forints?

(13 points)

Solution: a) The equation is

$$x^{\lg x} = 10^8 x^2$$

where $\lg$ denotes the base-10 logarithm. The logarithm is only defined for positive numbers, so $x > 0$. Both sides are positive, and the logarithm function is one-to-one, so we can take the base-10 logarithm of both sides:

$$\begin{aligned} \lg x^{\lg x} &= \lg\left(10^8 x^2\right) \\ \lg x \cdot \lg x &= \lg 10^8 + \lg x^2 \\ (\lg x)^2 - 2\lg x - 8 &= 0 \end{aligned}$$

With the new unknown $A = \lg x$ we get $A^2 - 2A - 8 = 0$, so $A_1 = 4$ and $A_2 = -2$. Therefore $x_1 = 10\,000$ and $x_2 = 0.01$.

b) When the interest is credited, $16\% + 6\% = 22\%$ of it is deducted, so the principal grows by only 78% of 2.5%, that is, by $0.78 \cdot 2.5\% = 1.95\%$. By the compound interest formula $T_n = T_0\left(1 + \frac{p}{100}\right)^n$, we look for the smallest n with

$$1\,200\,000 < 1\,000\,000\left(1 + \frac{1.95}{100}\right)^n$$

We divide both sides by one million and take the base-10 logarithm of both sides:

$$\begin{aligned} \lg 1.2 &< \lg 1.0195^n \\ \lg 1.2 &< n \cdot \lg 1.0195 \\ \frac{\lg 1.2}{\lg 1.0195} &< n \end{aligned}$$

Since $\frac{\lg 1.2}{\lg 1.0195} \approx 9.44$, our deposit exceeds 1.2 million forints at the end of the tenth year.

4. Consider the following function:

$$f(x) = \begin{cases} \dfrac{12}{x} & \text{if } x < -3 \\ \left|x^2 - 4\right| & \text{if } -3 \le x < 3 \\ -\dfrac{12}{x} & \text{if } x \ge 3 \end{cases}$$

a) Graph the function on the interval $[-12, 12]$.

b) Find the zeros of the function.

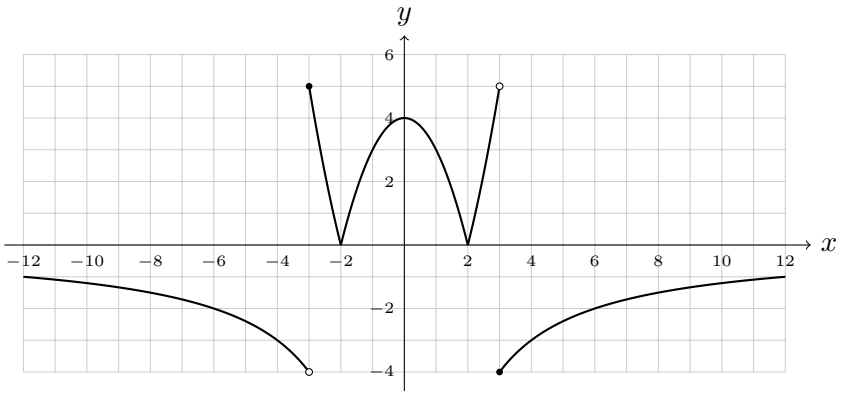
c) Find the range of the function.

d) Find the absolute maximum and minimum points and values of the function, and also its local maximum and minimum points and values.

e) Is f an even function?

(14 points)

Solution: a) See the figure.



b) The zeros of the function are $x_1 = -2$ and $x_2 = 2$. Outside the interval $[-3, 3]$ the function is never zero (the x-axis is a horizontal asymptote in both directions), so there are no other zeros.

c) The range of the function is $[-4, 5]$. (Careful: the question is about the range of the whole function, not only of the part that we graphed.)

d) The absolute maximum point is $x = -3$, the maximum value is 5. The absolute minimum point is $x = 3$, the minimum value is -4.

The local maximum point is $x = 0$, the local maximum value is 4. The local minimum points are $x = -2$ and $x = 2$, the local minimum value is 0.

e) $f(-3) = 5$ but $f(3) = -4$, so f is not an even function.

Part II

5. We have a tetrahedral die (a regular tetrahedron with uniform mass distribution, with the numbers 1, 2, 3, and 4 written on its faces) and an octahedral die (a regular octahedron with uniform mass distribution, with the natural numbers from 1 to 8 written on its faces). We roll the two dice together. The outcome of an experiment is the sum of the numbers written on the faces touching the table.

a) Draw a bar chart of the distribution of the random variable defined this way, and compute its expected value.

b) What is the probability that the number on the face of the octahedral die touching the table is greater than the number on the face of the tetrahedral die touching the table?

c) If we repeat the experiment many times, what is the probability that the outcome is 9 for the first time in the eighth experiment?

d) We did the experiment one hundred times. We recorded the outcomes and got the frequency table below. For which outcome is the absolute difference between the relative frequency and the theoretical probability, rounded to three decimal places, the largest?

outcome	2	3	4	5	6	7	8	9	10	11	12
frequency	0	6	10	13	15	8	16	14	11	5	2

(16 points)

Solution: a) The possible pairs (octahedral die; tetrahedral die):

1; 1	2; 1	3; 1	4; 1	5; 1	6; 1	7; 1	8; 1
1; 2	2; 2	3; 2	4; 2	5; 2	6; 2	7; 2	8; 2
1; 3	2; 3	3; 3	4; 3	5; 3	6; 3	7; 3	8; 3
1; 4	2; 4	3; 4	4; 4	5; 4	6; 4	7; 4	8; 4

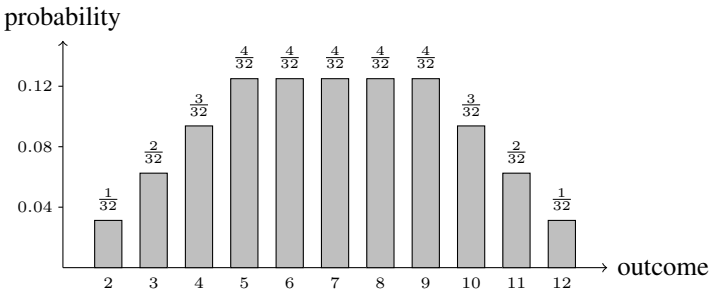
All 32 pairs are equally likely, so the probabilities of the possible outcomes are:

outcome	2	3	4	5	6	7	8	9	10	11	12
probability	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	$\frac{3}{32}$	$\frac{2}{32}$	$\frac{1}{32}$
rounded	0.031	0.063	0.094	0.125	0.125	0.125	0.125	0.125	0.094	0.063	0.031

The expected value is

$$2 \cdot \frac{1}{32} + 3 \cdot \frac{2}{32} + 4 \cdot \frac{3}{32} + 5 \cdot \frac{4}{32} + 6 \cdot \frac{4}{32} + 7 \cdot \frac{4}{32} + 8 \cdot \frac{4}{32} + 9 \cdot \frac{4}{32} + 10 \cdot \frac{3}{32} + 11 \cdot \frac{2}{32} + 12 \cdot \frac{1}{32} = \frac{224}{32} = 7$$

The bar chart of the distribution:



b) Counting in the table of pairs: when the tetrahedral die shows 1, 2, 3, or 4, the octahedral die is greater in 7, 6, 5, or 4 cases, respectively. So

$$P = \frac{\text{number of favorable cases}}{\text{number of all cases}} = \frac{7+6+5+4}{32} = \frac{22}{32} = \frac{11}{16}$$

c) From part a), $P\,(\text{the sum is } 9) = \frac{4}{32} = \frac{1}{8}$, so $P\,(\text{the sum is not } 9) = 1 - \frac{1}{8} = \frac{7}{8}$. The experiments are independent, so the probability in question is

$$P = \left(\frac{7}{8}\right)^7 \cdot \frac{1}{8} \approx 0.049$$

d) The relative frequencies:

outcome	2	3	4	5	6	7	8	9	10	11	12
frequency	0	6	10	13	15	8	16	14	11	5	2
relative frequency	0	0.06	0.10	0.13	0.15	0.08	0.16	0.14	0.11	0.05	0.02

Comparing them with the rounded theoretical probabilities, the largest absolute difference is $|0.08 - 0.125| = 0.045$, for the outcome 7.

6. a) The difference of the cubes of two positive integers is 169. What are these numbers?

b) Prove that if we subtract a positive integer from its fifth power, the difference is always divisible by each of the three smallest positive primes.

(16 points)

Solution: a) We look for the positive integer solutions of $x^3 - y^3 = 169$. The equation can be written as

$$(x-y)\left(x^2+xy+y^2\right) = 169$$

Here $x^2 + xy + y^2 > 0$, so $x - y$ must be positive. Since $169 = 13^2$, there are three cases.

Case I: $x - y = 1$ and $x^2 + xy + y^2 = 169$. Substituting $x = y + 1$, we get $3y^2 + 3y + 1 = 169$, that is, $y^2 + y - 56 = 0$, so $y = 7$ or $y = -8$. Only $y = 7$, $x = 8$ is a pair of positive integers.

Case II: $x - y = 13$ and $x^2 + xy + y^2 = 13$. Substituting $x = y + 13$, we get $3y^2 + 39y + 156 = 0$, that is, $y^2 + 13y + 52 = 0$, which has no real solutions.

Case III: $x - y = 169$ and $x^2 + xy + y^2 = 1$. For positive integers $x^2 + xy + y^2 \geq 3$, so there is no solution.

The two numbers are 8 and 7. (Check: $512 - 343 = 169$.)

b) The three smallest positive primes are 2, 3, and 5. Let n be a positive integer. We factor:

$$n^5 - n = n\left(n^4 - 1\right) = n\left(n^2 - 1\right)\left(n^2 + 1\right) = (n-1)\,n\,(n+1)\left(n^2+1\right)$$

One of the two consecutive numbers $n - 1$ and n is even, so the product is divisible by 2. One of the three consecutive numbers $n - 1$, n, $n + 1$ is divisible by 3, so the product is divisible by 3.

Divisibility by 5: if n leaves a remainder of 0, 1, or 4 when divided by 5, then n, $n - 1$, or $n + 1$ is divisible by 5. If $n = 5k + 2$ (k a nonnegative integer), then

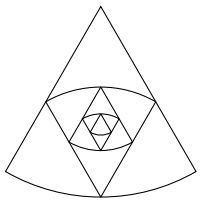
$$n^2 + 1 = (5k+2)^2 + 1 = 25k^2 + 20k + 5$$

is divisible by 5. If $n = 5k + 3$, then

$$n^2 + 1 = (5k+3)^2 + 1 = 25k^2 + 30k + 10$$

is divisible by 5. So in every case $n^5 - n$ is divisible by 2, 3, and 5.

7. We draw new, similar circular sectors into a circular sector of radius 1000 meters and central angle $60°$, as shown in the figure, and continue this "forever".



a) What are the area and the perimeter of the 12th inscribed (that is, the 13th drawn) sector?

b) Consider the sequence of the perimeters of the sectors, whose first term is the perimeter of the sector of radius 1000 meters. Compute the sum of its first 13 terms.

c) Consider the sequence of the areas of the sectors, whose first term is the area of the sector of radius 1000 meters, and the further sectors are obtained as described above. Denote these areas by T_1, T_2, and so on. What is the limit of $S_n = T_1 + T_2 + \cdots + T_n$ as $n \to \infty$?

(16 points)

Solution: a) Let A be the center of the big sector, C the midpoint of its arc (the center of the second sector), and B the point where the arc of the second sector meets a side of the big sector. Triangle ABC is isosceles, because its angles at A and at C are both $30°$; so its angle at B is $120°$. Let $AC = r_1$ and $BC = AB = r_2$. By the law of cosines in triangle ABC,

$$r_1^2 = r_2^2 + r_2^2 - 2r_2^2 \cos 120° = 3r_2^2, \quad \text{so} \quad r_2 = \frac{r_1}{\sqrt{3}}$$

So the radius of each new sector is the radius of the previous sector divided by $\sqrt{3}$. The radii (in meters) form a geometric sequence with first term $r_1 = 1000$ and ratio $q = \frac{1}{\sqrt{3}}$. Then

$$r_{13} = r_1 q^{12} = 1000 \cdot \left(\frac{1}{\sqrt{3}}\right)^{12} = \frac{1000}{3^6} = \frac{1000}{729}$$

The area of the 13th drawn sector is

$$T_{13} = r_{13}^2 \cdot \frac{\pi}{6} = \left(\frac{1000}{729}\right)^2 \cdot \frac{\pi}{6} = \frac{1\,000\,000\pi}{531\,441 \cdot 6} \approx 0.985 \text{ m}^2$$

and its perimeter (two radii and the arc) is

$$K_{13} = 2r_{13} + 2r_{13} \cdot \frac{\pi}{6} = 2r_{13}\left(1 + \frac{\pi}{6}\right) = \frac{2000}{729}\left(1 + \frac{\pi}{6}\right) \approx 4.18 \text{ m}$$

b) The perimeter of the nth sector is

$$K_n = 2r_n\left(1 + \frac{\pi}{6}\right) = 2r_1\left(1 + \frac{\pi}{6}\right) q^{n-1}$$

so the perimeters also form a geometric sequence, with first term $2000\left(1 + \frac{\pi}{6}\right)$ and ratio $q = \frac{1}{\sqrt{3}}$. The sum of its first 13 terms is

$$S_{13} = K_1 \cdot \frac{q^{13} - 1}{q - 1} = 2000\left(1 + \frac{\pi}{6}\right) \cdot \frac{\left(\frac{1}{\sqrt{3}}\right)^{13} - 1}{\frac{1}{\sqrt{3}} - 1} \approx 7204.04 \text{ m}$$

c) The area of the nth sector is

$$T_n = r_n^2 \cdot \frac{\pi}{6} = \left(r_1\left(\frac{1}{\sqrt{3}}\right)^{n-1}\right)^2 \cdot \frac{\pi}{6} = r_1^2 \cdot \frac{\pi}{6} \cdot \left(\frac{1}{3}\right)^{n-1}$$

So the areas form a geometric sequence with first term $T_1 = \frac{1000^2\pi}{6}$ and ratio $\frac{1}{3} < 1$. By the formula $\lim\limits_{n\to\infty} S_n = \frac{a_1}{1-q}$,

$$\lim_{n\to\infty} S_n = \frac{\frac{1000^2\pi}{6}}{1-\frac{1}{3}} = \frac{1000^2\pi}{6}\cdot\frac{3}{2} = 500^2\pi = 250\,000\pi \approx 785\,398 \text{ m}^2$$

8. The vertices of a triangle are $A\,(2,5)$, $B\,(7,-7)$, and $C\,(-4,-3)$. The interior angle bisector from vertex A intersects the side BC at the point D.

a) Find the equation of the interior angle bisector through A.

b) Find the coordinates and the length of the vector $\overrightarrow{AD}$.

c) Find the angle between the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

d) Can there be a triangle ABC in which the segment of the interior angle bisector from A inside the triangle is longer than the side AB? If yes, what is the condition for this, in terms of the angles of the triangle?

(16 points)

Solution: a) A direction vector of the interior angle bisector through A is the sum of the unit vectors in the directions of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = (5,-12)\,, \quad \left|\overrightarrow{AB}\right| = 13, \qquad \overrightarrow{AC} = (-6,-8)\,, \quad \left|\overrightarrow{AC}\right| = 10$$

$$\frac{1}{13}(5,-12) + \frac{1}{10}(-6,-8) = \left(-\frac{28}{130}, -\frac{224}{130}\right) = -\frac{28}{130}(1,8)$$

So $(1,8)$ is a direction vector and $(8,-1)$ is a normal vector of the bisector. The line passes through $A\,(2,5)$, so its equation is $8x - y = 8\cdot 2 - 5$, that is,

$$8x - y = 11 \tag{1}$$

b) D is the intersection of the bisector and the line BC. A direction vector of line BC is $\overrightarrow{BC} = (-11,4)$, so a normal vector is $(4,11)$, and the equation of line BC is

$$4x + 11y = -49 \tag{2}$$

From (1), $y = 8x - 11$. Substituting into (2): $4x + 88x - 121 = -49$, so $x = \frac{72}{92} = \frac{18}{23}$ and $y = \frac{144}{23} - 11 = -\frac{109}{23}$. So $D\left(\frac{18}{23}, -\frac{109}{23}\right)$ and

$$\overrightarrow{AD} = \left(\frac{18}{23} - 2, -\frac{109}{23} - 5\right) = \left(-\frac{28}{23}, -\frac{224}{23}\right), \qquad \left|\overrightarrow{AD}\right| = \frac{28}{23}\sqrt{1^2+8^2} = \frac{28\sqrt{65}}{23} \approx 9.81$$

c) For the angle φ between $\overrightarrow{AD}$ and $\overrightarrow{BC} = (-11,4)$:

$$\cos\varphi = \frac{\overrightarrow{AD}\cdot\overrightarrow{BC}}{\left|\overrightarrow{AD}\right|\cdot\left|\overrightarrow{BC}\right|} = \frac{\left(-\frac{28}{23}\right)(-11) + \left(-\frac{224}{23}\right)\cdot 4}{\frac{28\sqrt{65}}{23}\cdot\sqrt{137}} = \frac{-588}{28\sqrt{65}\sqrt{137}} = \frac{-21}{\sqrt{8905}} \approx -0.2225$$

so $\varphi \approx 102.86^\circ$.

d) Yes, there is such a triangle. Let the angles of the triangle be α, β, γ at A, B, C, and let $\varepsilon = \angle ADB$. In triangle ABD, the angle at A is $\frac{\alpha}{2}$, so $\varepsilon = 180^\circ - \beta - \frac{\alpha}{2}$. In a triangle, the longer side is opposite the larger angle. AD is opposite the angle β, and AB is opposite the angle ε, so

$$AD > AB \iff \beta > \varepsilon \iff \beta > 180^\circ - \beta - \frac{\alpha}{2} \iff \beta > 90^\circ - \frac{\alpha}{4}$$

For example, if $\alpha = 80^\circ$, $\beta = 75^\circ$, $\gamma = 25^\circ$, then $\varepsilon = 65^\circ < \beta$, so $AD > AB$.

9. In a class of 30 students, 11 boys and 8 girls play a musical instrument. 9 boys and 14 girls play sports competitively, and 13 boys attend the math club. Three of the girls do not play any sport at all, but they attend the math club.

a) How many boys and how many girls are there in the class?

b) Each of the 8 girls who play an instrument marked exactly three of the others as friends on a social networking site (and those marked her too). Illustrate such a network of friendships with a graph.

c) Five of the boys who play sports competitively play water polo. When we asked them, they told us that they are also registered on the social networking site mentioned in part b). Each of them told us how many of the other four are mutual friends with him on the site. The answers were 4, 4, 3, 2, 1. What can we conclude?

d) Another class of 30 students has only 13 boys. The class is preparing a dance performance for a show, in which 15 couples will dance. The necessary number of girls will dress as boys. The girls who dress as boys, and the couples, are chosen by drawing lots. In how many ways can the couples be formed?

e) What is the probability that, as a result of the drawing in part d), Mária Kis, a student in the class, dances with a girl dressed as a boy?

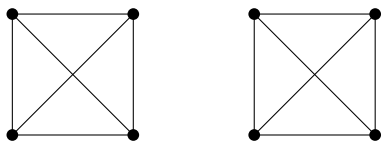
f) What is the probability that, as a result of the drawing in part d), at most two of the three competitive dancer girls, Mária Kis, Anita Nagy, and Karola Tóth, have to dress as boys?

g) In how many ways can eight different gifts be given out by drawing lots among the students of the class in part d), if each student can get at most one gift? In how many of these cases is there at least one boy among the winners?

(16 points)

Solution: a) There are 14 girls who play sports and 3 girls who do not play any sport, so there are at least 17 girls. There cannot be more, because there are at least 13 boys (the math club members). So there are 17 girls and 13 boys in the class.

b) We need a graph with 8 vertices in which every vertex has degree 3. For example, the following graph works: two groups of four girls, where everybody is friends with the other three girls of her group.



c) We would need a simple graph with 5 vertices with degrees 4, 4, 3, 2, 1. A vertex of degree 4 is connected to all other vertices. With two vertices of degree 4, every vertex is connected to both of them, so every vertex has degree at least 2. This contradicts the existence of a vertex of degree 1. So at least one of the five boys was mistaken.

d) Two girls have to dress as boys; they can be chosen from the 17 girls in $\binom{17}{2}$ ways. Then there are 15 "boys" and 15 girls, and the couples can be formed in 15! ways. So the couples can be formed in

$$\binom{17}{2} \cdot 15! \approx 1.78 \cdot 10^{14}$$

ways.

e) Favorable cases: Mária Kis does not dress as a boy, so the two girls dressing as boys can be chosen from the other 16 girls in $\binom{16}{2}$ ways. Mária's partner is one of these two girls (2 choices), and the other 14 couples can be formed in 14! ways. The number of all cases is known from part d). So

$$P = \frac{\binom{16}{2} \cdot 2 \cdot 14!}{\binom{17}{2} \cdot 15!} = \frac{120 \cdot 2}{136 \cdot 15} = \frac{2}{17}$$

f) Only two girls dress as boys, so it is certain that at most two of the three dancers have to dress as boys. The probability is 1.

g) Eight different gifts can be given to 8 of the 30 students, one gift each, in $\frac{30!}{22!} = 30 \cdot 29 \cdot \ldots \cdot 23$ ways. To count the cases with at least one boy among the winners, we subtract the number of cases with only girls among the winners from the number of all cases:

$$\frac{30!}{22!} - \frac{17!}{9!} = 235\,989\,936\,000 - 980\,179\,200 = 235\,009\,756\,800 \approx 2.35 \cdot 10^{11}$$

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