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Solutions: KoMaL website

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation over the set of real numbers.

$$\sqrt{x^2-5x-14}\cdot|5-x|\cdot\sin\left(2x+\frac{\pi}{6}\right)\cdot\lg(9-x)=0$$

(lg denotes the base 10 logarithm.)

(13 points)

Solution: Because of the logarithm, $9-x>0$, so $x<9$; because of the square root, $x^2-5x-14\geq 0$, so $x\leq -2$ or $x\geq 7$. Together, the domain is $x\leq -2$ or $7\leq x<9$.

A product is 0 exactly when one of its factors is 0.

$|5-x|=0$ gives $x=5$, but this is not in the domain. $\lg(9-x)=0$ gives $9-x=1$, so $x=8$, which is a solution. $\sqrt{x^2-5x-14}=0$ gives $x^2-5x-14=0$, so $x=-2$ and $x=7$ are solutions. $\sin\left(2x+\frac{\pi}{6}\right)=0$ gives $2x+\frac{\pi}{6}=k\pi$, where k is an integer, that is, $x=-\frac{\pi}{12}+k\frac{\pi}{2}$. Because of the domain, we get solutions only for $k<-1$ and for $k=5$. (For $k=-1$, $x\approx -1.83>-2$; for $k=5$, $x=\frac{29\pi}{12}\approx 7.59$; for $k=6$, $x\approx 9.16>9$.)

So the solution set is

$$\{-2,7,8\}\cup\left\{-\frac{\pi}{12}+k\frac{\pi}{2}:k\leq -2 \text{ or } k=5,\ k \text{ an integer}\right\}$$

and these numbers make the equation true.

2. a) How many digits does 5^{29} have in base ten? *(3 points)*

b) The first term of a geometric sequence is 5^{-29} and its common ratio is 5. At least how many terms, starting with the first one, do we have to multiply so that the product reaches 5^{29}? *(8 points)*

c) János Apáczai Csere was born exactly 400 years ago. Write the year of birth of the scholar in base five. *(2 points)*

Solution: a) Since $\lg 5^{29}=29\lg 5\approx 20.27$, that is, $20<\lg 5^{29}<21$, we have $10^{20}<5^{29}<10^{21}$. So 5^{29} has 21 digits.

b) Let P_n be the product of the first n terms of the geometric sequence:

$$P_n=a_1\cdot a_1q\cdot a_1q^2\cdots a_1q^{n-1}=a_1^n\cdot q^{1+2+\cdots+(n-1)}=a_1^n\cdot q^{\frac{n(n-1)}{2}}$$

With $a_1=5^{-29}$ and $q=5$, we need

$$\left(5^{-29}\right)^n\cdot 5^{\frac{n(n-1)}{2}}\geq 5^{29},\qquad \text{that is,}\qquad 5^{-29n+\frac{n(n-1)}{2}}\geq 5^{29}$$

The function 5^x is strictly increasing, so $-29n+\frac{n(n-1)}{2}\geq 29$, that is, $n^2-59n-58\geq 0$. For positive n this means $n\geq\frac{59+\sqrt{3713}}{2}\approx 59.97$. So we have to multiply at least 60 terms.

c) Apáczai was born in $2025-400=1625$. Since $1625=2\cdot 625+3\cdot 125=2\cdot 5^4+3\cdot 5^3$, in base five it is $23\,000_5$.

3. Nobody in a class of 34 students was born in February.

   a) Prove that there is a month in which at least four students of the class were born. *(3 points)*

   b) What is the probability that there are two students in this class who were born on the same day of the year? *(6 points)*

   c) The sorbet shop near the school sells ten flavors of fruit sorbet. At least how many times did the whole class have to go to the sorbet shop, if we can be sure that during these visits there were two exactly identical orders, if everyone can buy at most two scoops? (Two orders are different if the flavor of the lower or the upper scoop in the cone is different.) *(5 points)*

   Solution: a) If there were no four students born in the same month, then at most 3 students would be born in each of the 11 months, so the class would have at most $11 \cdot 3 = 33$ students. This is a contradiction, so there are four students born in the same month.

   b) If nobody was born in February, then $365 - 28 = 366 - 29 = 337$ possible days remain in a year. The number of cases in which everybody was born on a different day is $337 \cdot 336 \cdot 335 \cdots 304 = \frac{337!}{303!}$, and the number of all cases is 337^{34}. So the probability that everybody was born on a different day is

$$\frac{337 \cdot 336 \cdots 304}{337^{34}} \approx 0.1786$$

   and the probability in question is about $1 - 0.1786 = 0.821$.

   c) There are 10 different one-scoop orders. A two-scoop order can have two scoops of the same flavor (10 possibilities) or two different flavors (and the order matters), $10 \cdot 9 = 90$ possibilities. So there are 110 different orders. During one visit there are at most 34 orders, and since $3 \cdot 34 = 102 < 110 < 4 \cdot 34 = 136$, the class had to go at least 4 times. (After 3 visits, all 102 orders could still be different.)

   Note: In the original problem the shop sells ice cream; we changed it to fruit sorbet.

4. At a summer party of a class, some of the students buy food and drinks at the snack bar of the beach. Lujzi buys only one lángos (fried flatbread) for herself, but she also pays for the tea and fries menus of Sanyi, Szöszi, Kati, and Béla (four teas and four fries), so she pays 7130 Ft in all. Jani collected a few orders too: he buys eight teas, five lángos, and two fries, and pays 12 790 Ft. Armand invites only Zsuzsi and Amanda this time; he plans to buy a tea, a fries, and a lángos for each of the three of them, but he realizes that he would stay hungry, so he adds two more fries to the order. He pays with a 10 000 Ft bill and gets 230 Ft change.

   a) How much does a tea, a lángos, and a fries cost at this snack bar? *(8 points)*

   b) In how many different orders can all the food and drinks paid for above be handed out at the counter, if we do not distinguish between identical items? *(3 points)*

   Solution: The data in a table:

	lángos	tea	fries	total price (Ft)
Lujzi	1	4	4	7130
Jani	5	8	2	12 790
Armand	3	3	5	9770
total	9	15	11	

   a) Let the prices of a lángos, a tea, and a fries be l, t, and s Ft. Then

$$\begin{aligned} l + 4t + 4s &= 7130 \\ 5l + 8t + 2s &= 12\,790 \\ 3l + 3t + 5s &= 9770 \end{aligned}$$

Solving this system (by elimination or substitution), we get $l = 1210$, $t = 630$, and $s = 850$. So a lángos costs 1210 Ft, a tea 630 Ft, and a fries 850 Ft. (Check: $1210 + 2520 + 3400 = 7130$, $6050 + 5040 + 1700 = 12\,790$, and $3630 + 1890 + 4250 = 9770$.)

b) They bought 9 lángos, 15 teas, and 11 fries, 35 items in all. The number of possible orders (permutations with repetition) is

$$\frac{35!}{9! \cdot 15! \cdot 11!} \approx 5.455 \cdot 10^{14}$$

Note: Lángos is a Hungarian fried flatbread; we left out any toppings.

Part II

5. The vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{a} + \mathbf{b}| = 1$.

a) Find the dot product of the vectors $\mathbf{a}$ and $\mathbf{b}$. *(4 points)*

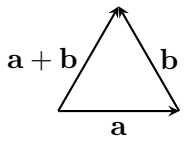
b) Find the length of the vector $2\mathbf{a} + \mathbf{b}$. *(4 points)*

Consider the vector $\mathbf{a}\,(3, 4)$. A tangent line of the circle of radius 6 centered at the origin that is parallel to the vector $\mathbf{a}$ meets the y-axis at the point $(0, p)$.

c) Find the possible values of p. *(8 points)*

Solution: a) The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}+\mathbf{b}$ can be drawn as the side vectors of an equilateral triangle, so the angle between $\mathbf{a}$ and $\mathbf{b}$ is $120°$. So their dot product is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos 120° = 1 \cdot 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$$



b) The vectors $2\mathbf{a}$, $\mathbf{b}$, and $2\mathbf{a} + \mathbf{b}$ can be drawn as the side vectors of a triangle with two sides 2 and 1 that form a $60°$ angle. Let $x = |2\mathbf{a} + \mathbf{b}|$. By the law of cosines in this triangle,

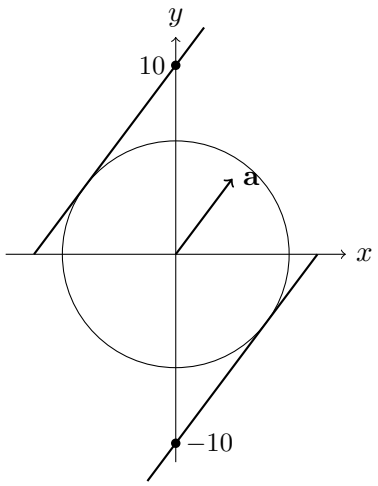
$$x^2 = 2^2 + 1^2 - 2 \cdot 2 \cdot 1 \cdot \cos 60° = 3, \qquad \text{so} \qquad x = \sqrt{3}$$

(Or: $|2\mathbf{a} + \mathbf{b}|^2 = 4\,|\mathbf{a}|^2 + 4\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = 4 - 2 + 1 = 3$.)

c) The slope of the line of the vector $\mathbf{a}$ is $\frac{4}{3}$, so the lines parallel to it are $y = \frac{4}{3}x + b$, where b is a real number. Such a line is a tangent line if it has exactly one common point with the circle. Substituting into the equation of the circle,

$$x^2 + \left(\frac{4}{3}x + b\right)^2 = 36, \qquad \text{that is,} \qquad 25x^2 + 24bx + 9b^2 - 324 = 0$$

There is exactly one common point when the discriminant is 0: $D = 576b^2 - 100\left(9b^2 - 324\right) = -324b^2 + 32\,400 = 0$, so $b = \pm 10$. The two tangent lines are $y = \frac{4}{3}x + 10$ and $y = \frac{4}{3}x - 10$, so the possible values of p are 10 and -10.



6. Mr. Zsigri, a teacher, takes part in a private pension savings plan: on the first banking day of 2001, and since then on the first banking day of every month, he paid in 40 000 Ft. The pension fund pays 0.4% interest per month all the time. To support saving for retirement, the state pays 5% of the amount paid in during the year, but at most 25 000 Ft per year, into the member's account on the first banking day of the next year.

   a) How much money will be in the teacher's account on the first banking day of 2026? (In 2026 the teacher no longer pays in, and the bank no longer computes interest; only the state credit for 2025 arrives.) *(7 points)*

   From part of the money collected by 2026, Mr. Zsigri goes on a trip around the world, but he leaves 12 million Ft in the account, so that he gets a constant monthly supplementary pension from it for exactly 20 years. (The fund credits the interest on the first day of every month, and then transfers the supplementary pension to the teacher's regular bank account free of charge.)

   b) What monthly supplementary pension can the teacher count on, if the fund continues to pay a fixed 0.4% monthly interest on the money in the account? *(6 points)*

   Each year, the probability that this pension fund lowers its interest rate is 0.002.

   c) What is the probability that the fund never lowers its interest rate during the whole 45 years of the process? *(3 points)*

   Solution: a) Each year the payments are $12 \cdot 40\,000 = 480\,000$ Ft in all, and the state credit is 5% of this, 24 000 Ft. Since every 40 000 Ft payment earns interest monthly, after one year the account holds

$$A_1 = \underbrace{40\,000 \cdot 1.004^{12} + 40\,000 \cdot 1.004^{11} + \cdots + 40\,000 \cdot 1.004}_{\text{payments in 2001}} + \underbrace{24\,000}_{\text{state credit}}$$

   Ft. In 25 years, 300 months pass. The teacher's payments and the state credits keep earning interest, but the state credits are added only every 12 months. So on the first banking day of 2026 the account holds

$$\begin{aligned} A_{25} &= 40\,000\left(1.004^{300} + 1.004^{299} + \cdots + 1.004\right) + 24\,000\left(1.004^{288} + 1.004^{276} + \cdots + 1\right) \\ &= 40\,000 \cdot 1.004 \cdot \frac{1.004^{300} - 1}{1.004 - 1} + 24\,000 \cdot \frac{\left(1.004^{12}\right)^{25} - 1}{1.004^{12} - 1} \approx 23\,214\,280 + 1\,130\,876 = 24\,345\,156 \end{aligned}$$

   Ft, using the sum formulas of two geometric sequences: the monthly payments with ratio 1.004 and 300 terms, and the state credits with ratio 1.004^{12} and 25 terms. So the teacher will have about 24.345 million Ft.

   b) Let x be the monthly payment and B_n the amount left in the account at the end of month n:

$$B_1 = 12 \cdot 10^6 \cdot 1.004 - x, \quad B_2 = \left(12 \cdot 10^6 \cdot 1.004 - x\right) \cdot 1.004 - x = 12 \cdot 10^6 \cdot 1.004^2 - x \cdot 1.004 - x$$

and similarly, at the end of the last, 240th month,

$$B_{240} = 12 \cdot 10^6 \cdot 1.004^{240} - x\left(1.004^{239} + 1.004^{238} + \cdots + 1.004 + 1\right)$$

The money runs out exactly, so $B_{240} = 0$. The expression in parentheses is the sum of the first 240 terms of a geometric sequence, so

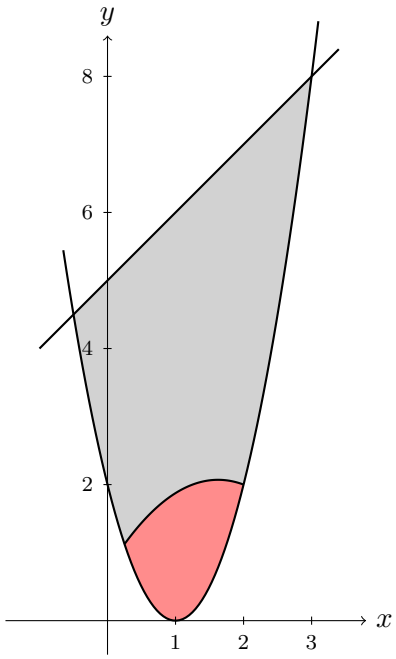
$$12 \cdot 10^6 \cdot 1.004^{240} = x \cdot \frac{1.004^{240} - 1}{1.004 - 1}, \qquad \text{so} \qquad x = 12 \cdot 10^6 \cdot 1.004^{240} \cdot \frac{1.004 - 1}{1.004^{240} - 1} \approx 77\,875$$

So the monthly supplementary pension is about 77 875 Ft.

c) The probability that the fund does not lower its interest rate in a given year is 0.998. So the probability that it does not lower it in any of the 45 years is $0.998^{45} \approx 0.914$.

7. A ghost train is making a sign in the shape of a monster's mouth. In the coordinate plane, the boundary lines of its design are given by the line $y = x + 5$ and the graphs of the functions

$$f(x) = 2x^2 - 4x + 2 \qquad \text{and} \qquad g(x) = -\frac{1}{2}x^2 + \frac{13}{8}x + \frac{3}{4}$$



How many deciliters of red paint are needed for the tongue, and how many deciliters of gray paint for the inside of the mouth, if 7 dl of paint is needed per square meter of both colors, and one unit on the coordinate axes of the design corresponds to 50 cm in reality? (The teeth will be a white LED lamp, but the area under it is also painted gray.) Round your answers to whole numbers.

(16 points)

Solution: f is a quadratic function with a minimum and g one with a maximum; their graphs are the boundaries of the mouth and of the tongue. The tongue meets the bottom of the mouth where $f(x) = g(x)$:

$$2x^2 - 4x + 2 = -\frac{1}{2}x^2 + \frac{13}{8}x + \frac{3}{4}, \qquad \text{that is,} \qquad \frac{5}{2}x^2 - \frac{45}{8}x + \frac{5}{4} = 0$$

so $x_1 = \frac{1}{4}$ and $x_2 = 2$. The slanted boundary line of the mouth meets the graph of f where $2x^2 - 4x + 2 = x + 5$, that is, at $x_3 = -\frac{1}{2}$ and $x_4 = 3$. So the area of the red tongue is

$$T_{\text{tongue}} = \int_{\frac{1}{4}}^{2} \left(-\frac{5}{2}x^2 + \frac{45}{8}x - \frac{5}{4} \right) dx = \left[-\frac{5}{6}x^3 + \frac{45}{16}x^2 - \frac{5}{4}x \right]_{\frac{1}{4}}^{2} = \frac{1715}{768} \approx 2.233$$

square units, and the area of the whole mouth is

$$\begin{aligned} T_{\text{mouth}} &= \int_{-\frac{1}{2}}^{3} \left((x+5) - \left(2x^2 - 4x + 2 \right) \right) dx = \int_{-\frac{1}{2}}^{3} \left(-2x^2 + 5x + 3 \right) dx \\ &= \left[-\frac{2}{3}x^3 + \frac{5}{2}x^2 + 3x \right]_{-\frac{1}{2}}^{3} = \frac{343}{24} \approx 14.292 \end{aligned}$$

square units. One unit square is $50 \text{ cm} \cdot 50 \text{ cm} = 2500 \text{ cm}^2 = 0.25 \text{ m}^2$ in reality. So the tongue is about $2.233 \cdot 0.25 \approx 0.558 \text{ m}^2$, the mouth is about $14.292 \cdot 0.25 \approx 3.573 \text{ m}^2$, and the gray area is about $3.573 - 0.558 = 3.015 \text{ m}^2$. So we need about $3.015 \cdot 7 \approx 21$ dl of gray paint and about $0.558 \cdot 7 \approx 4$ dl of red paint.

8. The base of an isosceles triangle is 10 units and its legs are 13 units long. We inscribe rectangles in the triangle with one side on the base of the triangle and the other two vertices on the two legs.

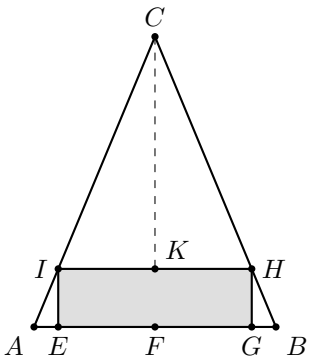
   a) Find the area of the rectangle whose side on the base of the triangle is 8 units long. *(5 points)*

   b) Find the sides of the inscribed rectangle of maximal area. *(5 points)*

   We rotate the triangle in two ways: A) about its base, B) about one of its legs.

   c) Find the exact value of the ratio of the surface areas of the solids we get in cases A) and B). *(6 points)*

   Solution: a) Let AB be the base of the isosceles triangle ABC, F the midpoint of AB, and $EGHI$ the inscribed rectangle, with E and G on the base, I on the leg AC, and K the point where HI meets CF.



The height CF of the triangle is an axis of symmetry of both the triangle and the rectangle, so $AF = FB = 5$ and $EF = FG = 4$. By the Pythagorean theorem in the triangle AFC, $CF = \sqrt{13^2 - 5^2} = 12$. The triangles AFC and IKC are similar (their corresponding sides are parallel), so

$$\frac{4}{5} = \frac{12 - KF}{12}, \qquad \text{so} \qquad KF = 2.4$$

So the area of the rectangle $EGHI$ is $8 \cdot 2.4 = 19.2$ square units.

b) Let the sides of the rectangle be $2x$ and y. As in part a), $\frac{x}{5} = \frac{12-y}{12}$, so $y = 12 - 2.4x$, and the area of the rectangle is

$$T(x) = 2x(12 - 2.4x)$$

This quadratic function has a maximum; its zeros are $x_1 = 0$ and $x_2 = \frac{12}{2.4} = 5$, so its maximum point is $\frac{x_1+x_2}{2} = 2.5$. So the sides of the rectangle of maximal area are $2x = 5$ and $y = 12 - 2.4 \cdot 2.5 = 6$ units.

Note: KoMaL writes the area as $x\,(12 - 2.4x)$, which is half of it; the maximum point is the same.

c) In case A) we get a double cone whose slant side is 13 and whose base radius is 12 (the height of the triangle). Its surface area is $A_A = 2 \cdot 12 \cdot 13\pi = 312\pi$.

In case B) we get a double cone whose base radius is the height m_b of the triangle to a leg, and whose slant sides are 10 and 13. (The triangle is acute, so the foot of this height is on the leg.) Writing the area of the triangle in two ways, $\frac{10 \cdot 12}{2} = \frac{13 \cdot m_b}{2}$, so $m_b = \frac{120}{13}$. So the surface area of this double cone is

$$A_B = \frac{120}{13} \cdot 10\pi + \frac{120}{13} \cdot 13\pi = \frac{2760}{13}\pi$$

and the ratio of the surface areas is

$$\frac{A_A}{A_B} = \frac{312\pi}{\frac{2760}{13}\pi} = \frac{169}{115}$$

9. In an inspection, they examined how much the mass of rice in 1 kg packages really differs from 1000 grams. They made two series of 8 measurements each; the results are in the table below.

measured differences in grams								
series 1	1	3	5.5	2.5	3	6	5	0
series 2	3	6	6	3	3	3		6

a) Find the mean and the standard deviation of the first series in grams, to one decimal place. *(3 points)*

b) Draw a box plot of the data of the first series. *(5 points)*

The standard deviation of the second series is 1.5 grams.

c) What could the missing data value in the table be? *(8 points)*

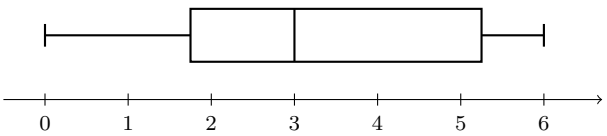
Solution: a) The mean is

$$\overline{x} = \frac{1 + 3 + 5.5 + 2.5 + 3 + 6 + 5 + 0}{8} = \frac{26}{8} = 3.25 \approx 3.3 \text{ grams}$$

The deviations from the mean are -2.25, -0.25, 2.25, -0.75, -0.25, 2.75, 1.75, and -3.25, so the standard deviation is

$$\sigma = \sqrt{\frac{2 \cdot 2.25^2 + 2 \cdot 0.25^2 + 0.75^2 + 2.75^2 + 1.75^2 + 3.25^2}{8}} = \sqrt{\frac{31.5}{8}} \approx 2.0 \text{ grams}$$

b) In increasing order, the data are 0, 1, 2.5, 3, 3, 5, 5.5, 6. The median is $Me = 3$, the lower quartile is $Q_1 = \frac{x_2+x_3}{2} = 1.75$, and the upper quartile is $Q_3 = \frac{x_6+x_7}{2} = 5.25$. The box plot is



c) Let the missing value be a and the mean of the second series b. Then $8b = 4 \cdot 3 + 3 \cdot 6 + a$, so $a = 8b - 30$. For the variance,

$$\frac{4\,(3-b)^2 + 3\,(6-b)^2 + (8b - 30 - b)^2}{8} = \left(\frac{3}{2}\right)^2$$

which leads to $56b^2 - 480b + 1026 = 0$. Its roots are $b_1 = \frac{9}{2}$ and $b_2 = \frac{57}{14}$, so the missing value is $a_1 = 8 \cdot \frac{9}{2} - 30 = 6$ grams or $a_2 = 8 \cdot \frac{57}{14} - 30 = \frac{18}{7}$ grams.

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