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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The arithmetic mean of two positive numbers is 205, and the difference of their arithmetic and geometric means is 160. Find the two numbers.

(11 points)

Solution: Let the two numbers be $x > 0$ and $y > 0$. Their arithmetic mean is

$$\frac{x+y}{2} = 205, \qquad \text{so} \qquad x + y = 410 \quad \text{and} \quad y = 410 - x$$

The geometric mean is at most the arithmetic mean, so by the condition it is 160 less than the arithmetic mean: $205 - 160 = 45$. Then

$$\sqrt{xy} = 45, \qquad \text{so} \qquad xy = 45^2 = 2025$$

Substituting $y = 410 - x$, we get the quadratic equation

$$x\,(410 - x) = 2025, \qquad \text{that is,} \qquad x^2 - 410x + 2025 = 0$$

with roots $x_1 = 405$ and $x_2 = 5$. Then $y_1 = 5$ and $y_2 = 405$, so we get the same pair of numbers twice. Check: the arithmetic mean of 5 and 405 is indeed 205, and their geometric mean is $\sqrt{2025} = 45$, which is 160 less. So the two numbers are 5 and 405.

2. a) Find the real number x, if $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$, where $A\,(x, 7)$, $B\,(4, -1)$, and $C\,(x - 11, -4)$. *(8 points)*

b) Find the centroid of the triangle ABC obtained in part a). *(2 points)*

c) Find the area of the triangle ABC obtained in part a). *(3 points)*

Solution: a) The coordinates of the two vectors are

$$\overrightarrow{AB} = (4 - x, -8) \qquad \text{and} \qquad \overrightarrow{AC} = (-11, -11)$$

so their dot product is

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = (4 - x)\,(-11) + (-8)\,(-11) = 0$$

which gives $x = -4$. Check: for $x = -4$, $\overrightarrow{AB} = (8, -8)$ and $\overrightarrow{AB} \cdot \overrightarrow{AC} = 8 \cdot (-11) + (-8)\,(-11) = -88 + 88 = 0$.

b) With $x = -4$, the vertices are $A\,(-4, 7)$, $B\,(4, -1)$, and $C\,(-15, -4)$. The coordinates of the centroid are the averages of the coordinates of the vertices:

$$S\left(\frac{-4 + 4 - 15}{3}, \frac{7 - 1 - 4}{3}\right) = S\left(-5, \frac{2}{3}\right)$$

c) By part a), $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular, so the triangle ABC has a right angle at A. Its legs are $AB = \sqrt{8^2 + (-8)^2} = 8\sqrt{2}$ and $AC = \sqrt{(-11)^2 + (-11)^2} = 11\sqrt{2}$, so its area is

$$T = \frac{8\sqrt{2} \cdot 11\sqrt{2}}{2} = 88$$

3. a) Find the range of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -3\cos(4x) + 2$. *(6 points)*

   b) Show by computation that the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2025x^3 - 2026x$ is odd. *(6 points)*

   Solution: a) The range of $\cos x$ is $[-1, 1]$, and so is the range of $\cos(4x)$. The range of $-3\cos(4x)$ is $[-3, 3]$, so the range of $f(x) = -3\cos(4x) + 2$ is $[-1, 5]$.

   b) For every real number x,

$$-g(-x) = -\left(2025(-x)^3 - 2026(-x)\right) = -\left(-2025x^3 + 2026x\right) = 2025x^3 - 2026x = g(x)$$

   So $g(-x) = -g(x)$ for every real x, that is, g is odd.

4. a) A rectangular plot of land lies along a straight river. The side parallel to the river is four times as long as the other side. There is no fence on the river side, and the total length of the fence along the other three sides is 600 meters. Find the area of the plot. *(3 points)*

   b) Another rectangular plot by the river is also fenced with 600 meters of fence, with no fence on the river side, and so that the area of the plot is as large as possible. Find the area of this plot. *(8 points)*

   c) How many divisors does $-45\,000$ have? *(4 points)*

   Solution: a) Let the shorter side be x meters; then the longer side is $4x$ meters. The fence is along the three sides other than the river side: $x + 4x + x = 6x = 600$, so $x = 100$ meters. The longer side is 400 meters, so the area of the plot is $100 \cdot 400 = 40\,000$ m^2.

   b) Let the side perpendicular to the river be y meters and the side parallel to the river z meters. Then the fence is $2y + z = 600$ meters long, so $z = 600 - 2y$. The area as a function of y is

$$T(y) = y(600 - 2y) = 600y - 2y^2 = -2(y - 150)^2 + 45\,000$$

   This downward opening parabola has its maximum at $y = 150$, so the largest possible area is $45\,000$ m^2. Check: if $y = 150$ m, then $z = 300$ m, and the area is indeed $150 \cdot 300 = 45\,000$ m^2.

   c) The prime factorization is $45\,000 = 2^3 \cdot 3^2 \cdot 5^4$, so the number of positive divisors is $(3 + 1)(2 + 1)(4 + 1) = 60$. The opposite of every positive divisor is also a divisor, so $-45\,000$ has $2 \cdot 60 = 120$ divisors.

Part II

5. a) Consider the line $e : x - y = 8$ and the circle $k : x^2 + y^2 - 6x + 4y + 4 = 0$. Determine the relative position of the line e and the circle k, and find their common points, if there are any. *(8 points)*

   b) Consider the points $P\,(3, -5)$ and $Q\,(6, -2)$ in the coordinate plane. Let P_1 and Q_1 be the perpendicular projections of P and Q onto the x-axis, and P_2 and Q_2 their perpendicular projections onto the y-axis. Find the perimeter of the quadrilateral $Q_1P_1Q_2P_2$. *(5 points)*

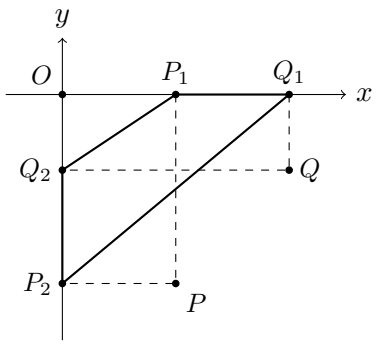
   c) Find the area of the quadrilateral $Q_1P_1Q_2P_2$. *(3 points)*

   Solution: a) We solve the system formed by the two equations. From the equation of the line, $y = x - 8$; substituting this into the equation of the circle,

$$x^2 + (x-8)^2 - 6x + 4\,(x-8) + 4 = 0, \qquad \text{that is,} \qquad 2x^2 - 18x + 36 = 0$$

   so $x_1 = 3$ and $x_2 = 6$, and then $y_1 = -5$ and $y_2 = -2$. The common points are $M_1\,(3, -5)$ and $M_2\,(6, -2)$, so the line e is a secant of the circle k.

   b) Projecting a point onto the x-axis keeps its first coordinate and makes the second coordinate 0, so $P_1\,(3, 0)$ and $Q_1\,(6, 0)$. Projecting onto the y-axis makes the first coordinate 0, so $P_2\,(0, -5)$ and $Q_2\,(0, -2)$.



   The sides of the quadrilateral $Q_1P_1Q_2P_2$ are

$$Q_1P_1 = |6-3| = 3, \quad P_1Q_2 = \sqrt{3^2 + 2^2} = \sqrt{13}, \quad Q_2P_2 = |-2+5| = 3, \quad P_2Q_1 = \sqrt{6^2 + 5^2} = \sqrt{61}$$

   so its perimeter is $K = 6 + \sqrt{13} + \sqrt{61}$.

   c) Let O be the origin. The area of the quadrilateral is the difference of the areas of the right triangles Q_1OP_2 and P_1OQ_2:

$$T = \frac{6 \cdot 5}{2} - \frac{3 \cdot 2}{2} = 15 - 3 = 12$$

6. a) Anna, Bogi, and Cili are cousins. In six years Bogi will be as old as Anna is now. In seven years Anna will be 9 times as old as Cili is now. In eight years the three of them together will be 42 years old. How old are the girls now? *(10 points)*

   b) Solve the inequality $\left(\frac{2}{5}\right)^{\ln x} \geq 6.25$ over the set of real numbers. *(6 points)*

   Solution: a) Let Anna's present age be a, Bogi's b, and Cili's c. In six years Bogi will be as old as Anna is now, so $b = a - 6$. In seven years Anna will be 9 times as old as Cili is now, so $9c = a + 7$, that is, $c = \frac{a+7}{9}$. In eight years they will be 42 years old together, so $(a+8) + (b+8) + (c+8) = 42$, that is, $a + b + c = 18$. Substituting,

$$a + (a-6) + \frac{a+7}{9} = 18, \qquad 18a - 54 + a + 7 = 162$$

   so $a = 11$. Then $b = a - 6 = 5$ and $c = \frac{11+7}{9} = 2$. So Anna is 11, Bogi is 5, and Cili is 2 years old.

   Check: in six years Bogi will be 11, as old as Anna is now. In seven years Anna will be 18, which is 9 times 2. In eight years they will be $19 + 13 + 10 = 42$ years old together.

   b) The domain is $x > 0$. We write 6.25 as a power of $\frac{2}{5}$:

$$\left(\frac{2}{5}\right)^{\ln x} \geq \left(\frac{2}{5}\right)^{-2}$$

   Since $\frac{2}{5} < 1$, a smaller exponent gives a larger power, so $\ln x \leq -2 = \ln e^{-2}$. The function $\ln x$ is strictly increasing, so $x \leq e^{-2}$. With the domain $x > 0$, the solution set is

$$0 < x \leq e^{-2}$$

   (We used only equivalent transformations.)

7. a) Prove that the decimal number $2.\dot{0}2\dot{5} = 2.025025025\ldots$ is rational. *(6 points)*

   b) Show that $2 \lg 7$ is irrational. ($\lg$ denotes the base 10 logarithm.) *(5 points)*

   c) Compute the indefinite integral $\int \left(5x^2 - 3x^9 + 8x\right) dx$. *(5 points)*

   Solution: a)

$$2.\dot{0}2\dot{5} = 2 + 0.025 + 0.000025 + \cdots = 2 + 0.025 + \frac{1}{10^3} \cdot 0.025 + \frac{1}{10^6} \cdot 0.025 + \cdots$$

   so the fractional part of the number is the sum of the geometric series with first term $a_1 = 0.025$ and ratio $q = \frac{1}{1000}$:

$$\frac{a_1}{1-q} = \frac{0.025}{1 - \frac{1}{1000}} = \frac{0.025}{\frac{999}{1000}} = \frac{25}{999}$$

   So $2.\dot{0}2\dot{5} = 2 + \frac{25}{999} = \frac{2023}{999}$, a quotient of two integers, so it is rational.

   b) Indirectly, suppose that $2 \lg 7$ is rational. Since $\lg 7$ is positive, it can be written as a quotient of positive integers p and q:

$$2 \lg 7 = \frac{p}{q}, \qquad \text{so} \qquad 2q \lg 7 = p, \quad \lg 7^{2q} = p, \quad 7^{2q} = 10^p = 2^p \cdot 5^p$$

   The left side is a power of 7 with a positive integer exponent. By the fundamental theorem of arithmetic, it cannot be equal to a number whose prime factorization does not contain 7, like the right side. This is a contradiction, so $2 \lg 7$ is irrational.

   c)

$$\int \left(5x^2 - 3x^9 + 8x\right) dx = \frac{5x^3}{3} - \frac{3x^{10}}{10} + 4x^2 + C, \qquad C \in \mathbb{R}$$

8. Consider the functions $f(x) = x^3$ and $g(x) = 2x + 1$, defined on the set of real numbers. Give the rules and the derivatives of the following composite functions.

   a) $(f \circ g)(x)$ *(2 points)*

   b) $(f \circ g)'(x)$ *(3 points)*

   c) $(f \circ f)(x)$ *(2 points)*

   d) $(f \circ f)'(x)$ *(2 points)*

   e) $(g \circ f)(x)$ *(2 points)*

   f) $(g \circ f)'(x)$ *(2 points)*

   g) $(g \circ g)(x)$ *(2 points)*

   h) $(g \circ g)'(x)$ *(1 points)*

   Solution: a) $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^3$

   b) $(f \circ g)'(x) = 3(2x+1)^2 \cdot 2 = 6(2x+1)^2$

   c) $(f \circ f)(x) = f\left(x^3\right) = \left(x^3\right)^3 = x^9$

   d) $(f \circ f)'(x) = 9x^8$

   e) $(g \circ f)(x) = g\left(x^3\right) = 2x^3 + 1$

   f) $(g \circ f)'(x) = 6x^2$

   g) $(g \circ g)(x) = g(2x+1) = 2(2x+1) + 1 = 4x + 3$

   h) $(g \circ g)'(x) = 4$

9. a) Töhötöm drew the interval $[-5, 8]$ on the number line, and with his eyes closed he pointed at one of its points. What is the probability that he pointed at a point of the interval $[-1, 2]$, if the probability is proportional to the length of the interval? *(3 points)*

   b) Prove that the intervals $[-5, 8]$ and $[-1, 2]$ have the same cardinality. *(5 points)*

   c) In how many ways can we choose exactly two numbers from the first five rows of Pascal's triangle so that their sum is even? (Two choices are different if at least one of the numbers is taken from a different place in Pascal's triangle.) *(4 points)*

   d) In how many ways can we choose at least seven numbers from the first five rows of Pascal's triangle so that their product is odd? (Two choices are different if at least one of the numbers is taken from a different place in Pascal's triangle.) *(4 points)*

   Solution: a) By the geometric probability model,

$$P = \frac{\text{favorable length}}{\text{total length}} = \frac{2-(-1)}{8-(-5)} = \frac{3}{13}$$

   b) Two sets have the same cardinality if there is a one-to-one correspondence between them. Let, for example, $f(x) = \frac{3}{13}x + \frac{2}{13}$. Then $f(-5) = -1$ and $f(8) = 2$, so f maps the interval $[-5, 8]$ onto the interval $[-1, 2]$. f is strictly increasing, so it is a bijection. So the two intervals have the same cardinality.

   c) The first five rows of Pascal's triangle are rows 0 through 4:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

There are $1+2+3+4+5=15$ numbers here; 4 of them are even and 11 are odd. The sum of two integers is even exactly when they have the same parity. Two even numbers can be chosen in $\binom{4}{2}=6$ ways, and two odd numbers in $\binom{11}{2}=55$ ways, so the two numbers can be chosen in $6+55=61$ ways.

d) A product of integers is odd exactly when every factor is odd. There are 11 odd numbers, and we choose 7, 8, 9, 10, or 11 of them, so the number of choices is

$$\binom{11}{7}+\binom{11}{8}+\binom{11}{9}+\binom{11}{10}+\binom{11}{11}=330+165+55+11+1=562$$

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