author: László Németh (Fonyód)
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Solutions: KoMaL website

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equations over the set of real numbers.

   a) $\frac{x}{x-1}+\frac{2x+1}{x+1}=\frac{3x+5}{x^2-1}$ *(5 points)*

   b) $\cos 2x+2\sin x+3=0$ *(5 points)*

   Solution: a) $x\neq\pm1$. Multiplying both sides by $(x-1)(x+1)=x^2-1$:

   $$x(x+1)+(x-1)(2x+1)=3x+5$$

   After rearranging and simplifying, $x^2-x-2=0$, with roots $x_1=-1$ and $x_2=2$. Because of the domain, the only solution is $x=2$. Check: $\frac{2}{1}+\frac{5}{3}=\frac{11}{3}$ and $\frac{6+5}{3}=\frac{11}{3}$.

   b) Using $\cos 2x=1-2\sin^2 x$: $1-2\sin^2 x+2\sin x+3=0$, that is, $\sin^2 x-\sin x-2=0$. So $\sin x=-1$ or $\sin x=2$; the latter is impossible. So the solutions are

   $$x=-\frac{\pi}{2}+2k\pi, \qquad \text{where } k \text{ is an integer}$$

   Check: for these values $\cos 2x=-1$ and $\sin x=-1$, so $-1-2+3=0$.

2. Consider the functions $f(x)=|x+1|$ $(x\in\mathbb{R})$ and $g(x)=\sqrt{x}-2$ $(x\in\mathbb{R},\ x\geq 0)$.

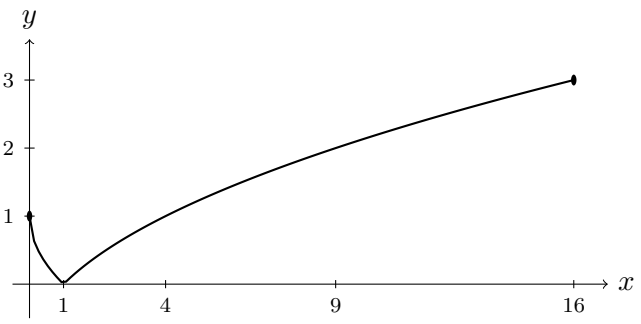
   a) Find the domain and the rule of the inverse of the function g. *(4 points)*

   b) Graph the function $f\circ g$ on the interval $[0,16]$. ($f\circ g$ is the function $f(g(x))$.) *(5 points)*

   c) Find all zeros of the function $g\circ f$. *(4 points)*

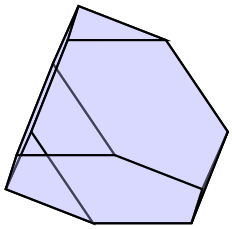
   Solution: a) We switch the variables: $x=\sqrt{y}-2$, $y\geq 0$, so $x\geq -2$ and $y=(x+2)^2$. So the inverse of g is $g^{-1}(x)=(x+2)^2$, with domain $x\geq -2$.

   b) $(f\circ g)(x)=|\sqrt{x}-2+1|=|\sqrt{x}-1|$ for $x\geq 0$.



c) $(g \circ f)(x) = \sqrt{|x+1|} - 2$. It is zero when $\sqrt{|x+1|} = 2$, that is, $|x+1| = 4$, so $x_1 = -5$ and $x_2 = 3$. Check: $\sqrt{|-5+1|} - 2 = \sqrt{4} - 2 = 0$ and $\sqrt{|3+1|} - 2 = 0$.

3. From a regular tetrahedron, we cut off a corner with the plane through the points of the three edges from that vertex that are one third of the edge length away from the vertex. We did this at every vertex, and we got the polyhedron in the figure, with all edges 10 cm long.



a) Find the surface area and the volume of the polyhedron. *(9 points)*

b) What angle does a hexagonal face form with a neighboring triangular face? *(5 points)*

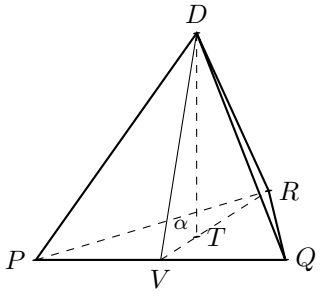
Solution: a) The surface consists of four regular hexagons and four equilateral triangles. A regular hexagon can be cut into six equilateral triangles like the ones at the corners, so the surface consists of $6 \cdot 4 + 4 = 28$ equilateral triangles with sides 10 cm. The area of an equilateral triangle with side a is $\frac{a^2\sqrt{3}}{4}$, so

$$A = 28 \cdot \frac{10^2\sqrt{3}}{4} = 700\sqrt{3} \approx 1212.4 \text{ cm}^2$$

Before the cuts, the solid was a regular tetrahedron with edges 30 cm. The volume of a regular tetrahedron with edge a is $\frac{\sqrt{2}}{12}a^3$. We subtract the volumes of the four small tetrahedra cut off (edges 10 cm):

$$V = \frac{\sqrt{2}}{12}\left(30^3 - 4 \cdot 10^3\right) = \frac{23\,000\sqrt{2}}{12} \approx 2710.6 \text{ cm}^3$$

b) The angle in question is the supplement of the angle α between two faces of a regular tetrahedron. Let PQR be a face of a regular tetrahedron $PQRD$, let V be the midpoint of the edge PQ, and T the center of the face PQR, which is the foot of the height from D.



VD and VR are heights of the congruent equilateral triangles PQD and PQR, so $VD = VR$. VR is also a median of the triangle PQR, and T (the centroid) divides it in the ratio $2:1$, so $VT = \frac{1}{3}VR = \frac{1}{3}VD$. In the right triangle VTD, the angle at V is the angle α between the faces PQR and PQD, and

$$\cos\alpha = \frac{VT}{VD} = \frac{1}{3}, \qquad \text{so} \qquad \alpha \approx 70.53^\circ$$

A hexagonal face and a neighboring triangular face form the angle $180^\circ - 70.53^\circ = 109.47^\circ$.

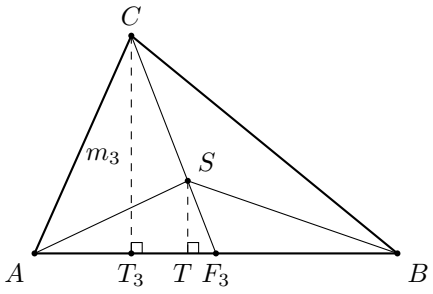
4. Decide whether the following statements A), B), and C) are true or false. Justify your answers.

   A) The segments connecting the centroid of a triangle to its vertices divide the triangle into three smaller triangles of equal area. *(6 points)*

   B) There is no quadratic equation with rational coefficients that has one rational root and one irrational root. *(4 points)*

   C) If we choose 30 days of a given year, it is not certain that there are at least five among them that fall on the same day of the week. *(4 points)*

   Solution: A) The statement is true. Let S be the centroid of the triangle ABC, F_3 the midpoint of AB, $CT_3 = m_3$ the height from C, and ST the height of the triangle ABS from S.



   We show that the area of the triangle ABS is one third of the area of the triangle ABC. S divides the median CF_3 in the ratio $2:1$, so $SF_3 = \frac{1}{3}CF_3$. The triangles CT_3F_3 and STF_3 are similar (their angles are equal), so the height ST of the triangle ABS to the side c is one third of m_3. So the area of ABS is one third of the area of ABC. The same holds for the other two small triangles, so their areas are equal.

   B) The statement is true. Indirectly, suppose that there is such an equation $ax^2 + bx + c = 0$ with a, b, c rational, $a \neq 0$, and roots x_1 rational and x_2 irrational. Then $x_1 + x_2 = -\frac{b}{a}$ is rational, and subtracting the rational number x_1 from it, we get a rational number, which cannot be equal to the irrational number x_2. This is a contradiction, so the statement is true.

   C) The statement is false.

   First solution: Indirectly, suppose that there are at most 4 days among the 30 for every day of the week. Then we could choose at most $4 \cdot 7 = 28$ days, but we chose 30, a contradiction.

   Second solution: By the general pigeonhole principle, if we put at least $nk + 1$ balls into n boxes, then some box contains more than k balls. Here the boxes are the 7 days of the week. Since $30 \geq 4 \cdot 7 + 1$, some day of the week occurs more than 4 times, that is, at least 5 times.

Part II

5. The first term of the sequence a_n is $a_1 = 1$, and from the second term on, every term can be computed with the recursive formula $a_{n+1} = a_n + 2n$.

   a) Prove that $a_n = n^2 - n + 1$. *(3 points)*

   b) Find the median and the lower and upper quartiles of the first 11 terms of the sequence. *(2 points)*

   c) Prove (by induction or another method) that the sum of the first n terms of the sequence is $S_n = \frac{n(n^2+2)}{3}$. *(6 points)*

   We choose a positive integer $n < 1000$, and compute the average of the first n terms of the sequence.

   d) What is the probability that the average is an integer, if every positive integer $n < 1000$ is chosen with the same probability? *(5 points)*

Solution: a)

$$\begin{aligned} a_n &= a_{n-1} + 2(n-1) = a_{n-2} + 2(n-2) + 2(n-1) = \cdots = a_1 + 2(1 + 2 + \cdots + (n-1)) \\ &= 1 + 2 \cdot \frac{(n-1)\,n}{2} = n^2 - n + 1 \end{aligned}$$

b) The first 11 terms are 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111. The median is 31, the lower quartile is 7, and the upper quartile is 73.

c) We use induction. First, $S_1 = 1 = \frac{1\cdot(1+2)}{3}$. If $S_k = \frac{k\left(k^2+2\right)}{3}$ (induction hypothesis), then

$$S_{k+1} = S_k + a_{k+1} = \frac{k\left(k^2+2\right)}{3} + (k+1)^2 - (k+1) + 1 = \frac{k^3+3k^2+5k+3}{3}$$

and on the other hand,

$$\frac{(k+1)\left((k+1)^2+2\right)}{3} = \frac{(k+1)\left(k^2+2k+3\right)}{3} = \frac{k^3+3k^2+5k+3}{3}$$

The two expressions are equal, which completes the proof.

Another method: since $a_n = n^2 - n + 1$, the sum is the sum of the first n squares, minus the sum of the first n positive integers, plus n times 1:

$$S_n = \frac{n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{6} + \frac{6n}{6} = \frac{n\left(2n^2+3n+1-3n-3+6\right)}{6} = \frac{n\left(n^2+2\right)}{3}$$

d) The average is $\frac{S_n}{n} = \frac{n^2+2}{3}$, which is an integer exactly when n^2+2 is divisible by 3. If n leaves remainder 1, 2, or 0 when divided by 3, then n^2 leaves remainder 1, 1, or 0, and n^2+2 leaves remainder 0, 0, or 2. So n^2+2 is divisible by 3 exactly when n is not divisible by 3. Of the positive integers less than 1000, $\frac{999}{3} = 333$ are divisible by 3 and 666 are not. So the probability is

$$P = \frac{666}{999} = \frac{2}{3}$$

6. The set H consists of the simple graphs whose five vertices are the vertices A, B, C, D, E of a given convex pentagon.

   a) How many elements does H have? *(3 points)*

   b) Draw all graphs in H with three edges that are pairwise not isomorphic. *(5 points)*

   c) How many graphs in H are connected and contain a cycle? *(8 points)*

   Solution: a) The complete graph on five vertices has $\binom{5}{2} = 10$ edges. For each of them we can decide independently whether it is in the graph or not, so there are $2^{10} = 1024$ such graphs.

   Or: a graph can contain 0, 1, 2, …, 10 of the 10 edges, so the number of graphs is

$$\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \cdots + \binom{10}{10} = 2^{10} = 1024$$

   b) The sum of the degrees is $2\cdot 3 = 6$, and a degree is at most 3. Listing the degrees in increasing order, $0+0+0+3+3$ and $0+0+1+2+3$ are not possible for a simple graph. Checking the other possibilities, we find exactly the following graphs:



$0+1+1+1+3$ $0+0+2+2+2$ $0+1+1+2+2$ $1+1+1+1+2$

There are no other possibilities, so H contains four pairwise non-isomorphic graphs with three edges.

Note: KoMaL's solution lists the last degree sequence as $1+1+1+1+1+2$ *(six terms); it is* $1+1+1+1+2$.

c) A simple graph on five vertices with fewer than four edges cannot be connected, and every graph with seven or more edges is connected and contains a cycle. So we have to examine the graphs with 4, 5, and 6 edges.

If a graph in H with four edges is connected, then it is a tree, so it contains no cycle.

Every graph in H with five edges contains a cycle, but not all of them are connected. A disconnected one must consist of an isolated vertex and a component of four vertices (with more than two components, or with components of 3 and 2 vertices, there could not be five edges). The four vertices can be chosen in 5 ways, and since the complete graph on four vertices has 6 edges, the 5 edges can be chosen in 6 ways. So $5 \cdot 6 = 30$ graphs with five edges are disconnected, and $\binom{10}{5} - 30 = 252 - 30 = 222$ are connected and contain a cycle.

A graph in H with six edges is disconnected only if one of its components is a complete graph on four vertices; there are 5 such graphs. So $\binom{10}{6} - 5 = 210 - 5 = 205$ are connected and contain a cycle.

So the number of connected graphs containing a cycle is

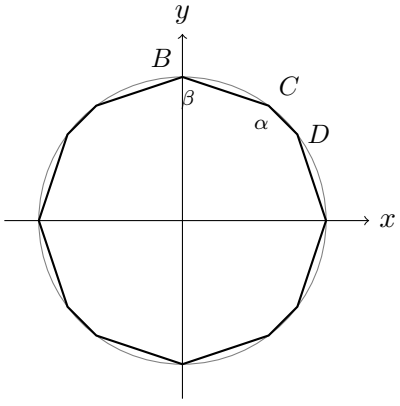
$$222 + 205 + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} = 222 + 205 + 120 + 45 + 10 + 1 = 603$$

7. The points with both coordinates integers on the circle of radius 5 centered at the origin determine an inscribed polygon.

a) Find the angles of the polygon. *(8 points)*

b) Find the second coordinates of the points of the y-axis from which the segment determined by the points $A\,(4,5)$ and $B\,(8,-3)$ is seen at an angle of 45°. *(8 points)*

Solution: a) The equation of the circle is $x^2 + y^2 = 25$; we look for its integer solutions. By the symmetry of the equation, it is enough to find the solutions with $0 \le x \le y$; the others come from changing signs and switching the coordinates. So the polygon is symmetric about the coordinate axes and the lines $y = \pm x$. If $0 \le x \le y$, then $2x^2 \le 25$, so $x = 0$, 1, 2, or 3, and we get an integer y only for $x = 0$, $y = 5$ and $x = 3$, $y = 4$. So the polygon is a 12-gon with vertices $(0, \pm 5)$, $(\pm 5, 0)$, $(\pm 3, \pm 4)$, and $(\pm 4, \pm 3)$.



By the symmetry, the angles at the vertices inside the quadrants (like C and D) are all equal (α), and the angles at the vertices on the axes are also equal (β). So we only need to compute two angles. With $B\,(0,5)$ and $C\,(3,4)$, $\overrightarrow{BC} = (3,-1)$, so

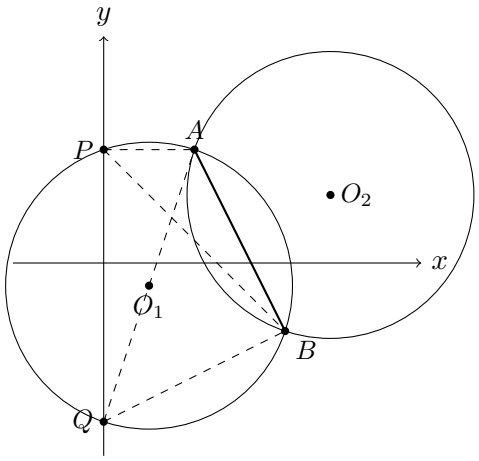
$$\tan\frac{\beta}{2} = \frac{3}{1}, \qquad \text{so} \qquad \beta = 2\tan^{-1} 3 \approx 143.130^\circ$$

By the symmetry about the line $y = x$, $\alpha = 45^\circ + 90^\circ + \left(90^\circ - \frac{\beta}{2}\right) \approx 153.435^\circ$. So the 12-gon has 4 angles of about 143.130° and 8 angles of about 153.435°. (Check: $4 \cdot 143.130^\circ + 8 \cdot 153.435^\circ \approx 1800^\circ = 10 \cdot 180^\circ$.)

b) The points from which a given segment is seen at a given angle form two arcs of circles, and the central angle of these arcs over the segment is twice the given angle (inscribed angle theorem). Here the angle is $45°$, so the central angle is $90°$: the centers of the arcs are the intersections of the Thales circle of AB and the perpendicular bisector of AB. These two points form a square with A and B.

The midpoint of AB is $F\,(6,1)$, and $\overrightarrow{FA} = (-2,4)$. Rotating $\overrightarrow{FA}$ by $+90°$, we get $(-4,-2)$, and $(6,1)+(-4,-2) = (2,-1)$; so one center is $O_1\,(2,-1)$. Rotating by $-90°$, we get $(4,2)$ and the other center $O_2\,(10,3)$. The radius is $\left|\overrightarrow{O_1A}\right| = |(2,6)| = 2\sqrt{10}$, so the circles are

$$k_1 : (x-2)^2 + (y+1)^2 = 40, \qquad k_2 : (x-10)^2 + (y-3)^2 = 40$$



The points of the y-axis have first coordinate 0. Substituting $x = 0$, the first equation gives $(y+1)^2 = 36$, so $y_1 = -7$ and $y_2 = 5$; the second equation has no real solution. Both points are on the arc of k_1 on the same side of AB as O_1, so the segment is seen at $45°$ (not $135°$) from them. So the points are $P\,(0,5)$ and $Q\,(0,-7)$, and the second coordinates in question are 5 and -7.

8. a) Prove that the graph of the function $f\,(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{2}$ $(x \in \mathbb{R})$ has no common point with the x-axis. *(2 points)*

b) Find the global and local extreme points and extreme values of f. *(6 points)*

c) Find the area of the closed bounded region enclosed by the graph of f, the tangent line to the graph at the point with first coordinate $x_0 = 2$, and the two coordinate axes. *(8 points)*

Solution: a)

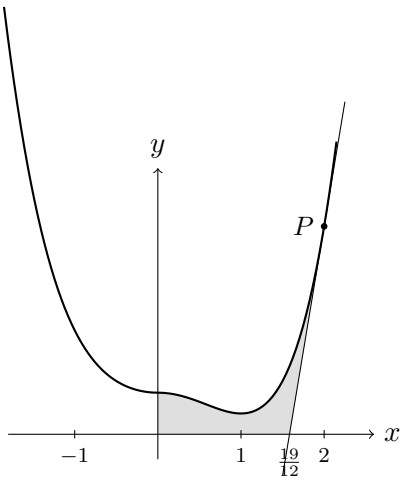
$$\frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{2} = \frac{1}{4}\left(x^4 - 2x^2 + 2\right) = \frac{1}{4}\left(\left(x^2-1\right)^2 + 1\right)$$

which is positive for every real number x. So the graph has no common point with the x-axis.

b) $f'\,(x) = x^3 - x$ and $f''\,(x) = 3x^2 - 1$. The zeros of $f'\,(x) = x\left(x^2-1\right)$ are $x_1 = 0$ and $x_{2,3} = \pm 1$; these are the possible extreme points. Since $f''\,(0) < 0$, f has a local maximum at 0, with value $\frac{1}{2}$. Since $f''\,(1) > 0$ and $f''\,(-1) > 0$, f has local minima at ± 1, both with value $\frac{1}{4}$. This is also the global minimum value, since $f\,(x) = \frac{1}{4}\left(x^2-1\right)^2 + \frac{1}{4} \geq \frac{1}{4}$. f has no global maximum, since $f\,(x)$ grows without bound as $x \to \pm\infty$.

Note: KoMaL's solution does not mention that there is no global maximum.

c) $f\,(2) = 2.5$, so the point of tangency is $P\,(2, 2.5)$. The slope of the tangent line is $f'\,(2) = 6$, so its equation is $y - 2.5 = 6\,(x-2)$, that is, $y = 6x - 9.5$. It meets the x-axis where $6x - \frac{19}{2} = 0$, at the point $\left(\frac{19}{12}, 0\right)$.



The area under the graph between 0 and 2 is

$$t_1 = \int_0^2 \left(\frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{2}\right) dx = \left[\frac{x^5}{20} - \frac{x^3}{6} + \frac{x}{2}\right]_0^2 = \frac{19}{15}$$

and the area under the tangent line between $\frac{19}{12}$ and 2 is

$$t_2 = \int_{\frac{19}{12}}^2 (6x - 9.5)\, dx = \left[3x^2 - 9.5x\right]_{\frac{19}{12}}^2 = \frac{25}{48}$$

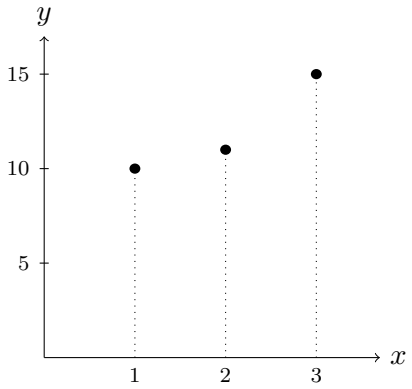
So the area of the region is $t = t_1 - t_2 = \frac{19}{15} - \frac{25}{48} = \frac{179}{240} \approx 0.75$ square units.

Remark: t_2 can also be computed without integration: it is the area of a right triangle with legs $2 - \frac{19}{12} = \frac{5}{12}$ and $\frac{5}{2}$, so $t_2 = \frac{1}{2} \cdot \frac{5}{12} \cdot \frac{5}{2} = \frac{25}{48}$.

9. In a survey about bottle return habits, a retail company asked 500 customers who had used a bottle return machine. At the machine, the refund can be requested as A) a store voucher, B) a bank transfer, or C) a charity donation. 398 customers chose A), 100 chose B), and 2 chose C).

   a) We choose 20 of the 500 customers at random. What is the probability that at most 16 of them chose option A)? Round your answer to four decimal places. *(5 points)*

   At one return point, $10\,000$ items were returned in the first week, $11\,000$ in the second week, and $15\,000$ in the third week. The figure shows these data; the unit on the vertical axis is a thousand items.



b) Give a linear function whose graph passes through the point $M\,(2, 12)$, and whose values at $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ have the smallest possible sum of squared differences from the values $y_1 = 10$, $y_2 = 11$, $y_3 = 15$. *(9 points)*

The returned items were sorted into three groups by material: a (plastic), b (metal), c (glass). Asked what kinds of bottles they usually return, 380 of the 500 customers above answered a, b, c; 50 answered a, b; 24 answered b, c; and 10 answered a, c. The customers who returned only one kind of item were divided among the types a, b, c in the ratio $3 : 2 : 1$.

c) How many customers returned bottles of type a? *(2 points)*

Solution: a) 102 customers did not choose A). The probability of the complement (at least 17 of the 20 chose A) is

$$P\left(\overline{E}\right) = \frac{\binom{398}{17}\binom{102}{3} + \binom{398}{18}\binom{102}{2} + \binom{398}{19}\binom{102}{1} + \binom{398}{20}}{\binom{500}{20}} \approx 0.3903$$

so the probability in question is $P\,(E) \approx 1 - 0.3903 = 0.6097$.

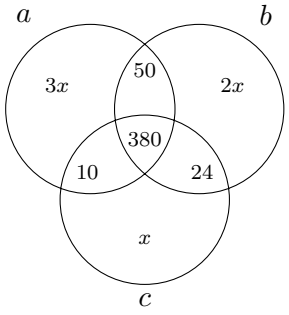
b) Let the linear function be $f\,(x) = mx + b$. Its graph passes through $M\,(2, 12)$, so $12 = 2m + b$ and $b = 12 - 2m$. At the given places, f takes the values $12 - m$, 12, and $12 + m$. Their differences from 10, 11, 15 are $2 - m$, 1, and $m - 3$, so we have to minimize

$$g\,(m) = (2 - m)^2 + 1^2 + (m - 3)^2 = 2m^2 - 10m + 14$$

Elementary solution: $g\,(m) = 2\,(m - 2.5)^2 + 1.5$ is smallest when $m = 2.5$. So the linear function is $f\,(x) = 2.5x + 7$.

With differentiation: $g'\,(m) = 4m - 10 = 0$ gives $m = 2.5$, and $g''\,(2.5) = 4 > 0$, so g has its minimum at $m = 2.5$.

c) Let $3x$, $2x$, and x be the numbers of customers who returned only type a, only type b, and only type c.



Adding the numbers in the regions of the Venn diagram,

$$380 + 50 + 24 + 10 + 6x = 500$$

so $x = 6$. So $380 + 50 + 10 + 3 \cdot 6 = 458$ customers returned bottles of type a (plastic).

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