author: Bálint Bíró
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Solutions: KoMaL website

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. a) Solve the following equation over the set of real numbers.

$$\cos\left(\frac{\pi}{6}\cos x\right) = \frac{\sqrt{3}}{2}$$

(5 points)

b) Find the positive prime numbers p and r in the equation $px + ry = 28$, if the pair of real numbers x, y is the solution of the system of equations $\sqrt{3x+y+5} = 4x - 20$, $x^2 + y = 29$. *(7 points)*

Solution: a) Let $\alpha = \frac{\pi}{6}\cos x$. The solutions of the equation $\cos\alpha = \frac{\sqrt{3}}{2}$ are

$$\alpha_1 = \frac{\pi}{6} + 2k\pi \qquad \text{and} \qquad \alpha_2 = \frac{11\pi}{6} + 2l\pi$$

where k and l are integers. So $\frac{\pi}{6}\cos x = \frac{\pi}{6} + 2k\pi$ or $\frac{\pi}{6}\cos x = \frac{11\pi}{6} + 2l\pi$. Dividing both equations by $\frac{\pi}{6}$, we get

$$\cos x = 1 + 12k \qquad \text{or} \qquad \cos x = 11 + 12l$$

Since $-1 \le \cos x \le 1$, only $k = 0$ is possible in the first equation and only $l = -1$ in the second one. So $\cos x = 1$ or $\cos x = -1$. The solutions are $x = m\pi$, where m is an integer; substituting them back, we can check that these are indeed solutions.

b) From the second equation, $y = 29 - x^2$. Substituting this into the first equation:

$$\sqrt{-x^2 + 3x + 34} = 4x - 20$$

Because of the square root, a solution must satisfy both $-x^2+3x+34 \ge 0$ and $4x-20 \ge 0$. By the quadratic formula, the first inequality holds exactly when $\frac{3-\sqrt{145}}{2} \le x \le \frac{3+\sqrt{145}}{2}$, that is, approximately $-4.52 \le x \le 7.52$. The second inequality holds for $x \ge 5$. So we look for x in the set

$$5 \le x \le \frac{3+\sqrt{145}}{2} \approx 7.52$$

Squaring both sides and rearranging, we get the quadratic equation $17x^2 - 163x + 366 = 0$, with solutions $x_1 = \frac{61}{17} \approx 3.59$ and $x_2 = 6$. x_1 is not in the set above, so it is not a solution. $x_2 = 6$ is in the set, and then $y = 29 - x^2 = -7$. We can check that $x = 6$, $y = -7$ is a solution of the system. So the equation $px + ry = 28$ becomes

$$6p - 7r = 28$$

p and r are positive primes, and the right side is even, so the left side is also even. This is only possible if $7r$ is even. So r is an even prime, $r = 2$, and then $p = 7$. The answer is $p = 7$, $r = 2$.

2. The following table lists the names, clubs, and numbers of goals of 11 of the top scorers of the 2024–2025 season of the men's handball Champions League, counting the goals scored in the season until November 17, 2024.

name	club	goals
Kamil Syprzak	Paris Saint-Germain	112
Dika Mem	Barcelona	106
Nedim Remili	Veszprém	99
Emil Madsen	GOG	98
Ómar Ingi Magnusson	Magdeburg	98
Mats Hoxer	Aalborg	96
Aaron Mensing	GOG	93
Niclas Ekberg	THW Kiel	92
Mario Sostaric	OTP-Bank Pick Szeged	88
Mitja Janc	Celje	87
Elohim Prandi	Paris Saint-Germain	87

a) Compute the range, the mean, and the standard deviation of the sample (the numbers of goals) in the table. *(3 points)*

b) Find the mode (or modes), the median, and the lower and upper quartiles of the numbers of goals, and draw the box plot. *(4 points)*

We add to the data the result of Ludovic Fabregas, who plays for Veszprém. His number of goals is the two-digit prime number p, which is the smallest element of the new data set of 12 numbers. If the lower quartile of the new data set is Q_1' and its upper quartile is Q_3', then we also know that $6.5\,(Q_3' - Q_1') < p < 8.5\,(Q_3' - Q_1')$.

c) How many goals could Ludovic Fabregas have scored in the Champions League until the given date? *(5 points)*

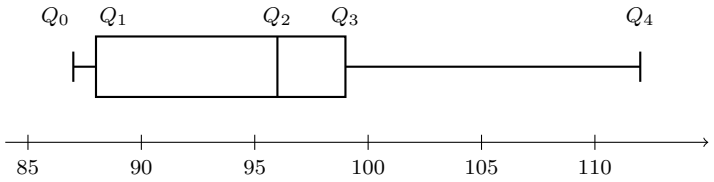
Solution: a) The range of the data is $R = 112 - 87 = 25$. The mean is

$$\overline{x} = \frac{112 + 106 + 99 + 98 + 98 + 96 + 93 + 92 + 88 + 87 + 87}{11} = 96$$

and the standard deviation is

$$\sigma = \sqrt{\frac{(112-96)^2 + (106-96)^2 + \cdots + (87-96)^2}{11}} = \sqrt{\frac{624}{11}} \approx 7.53$$

b) The data set has two modes, because 98 and 87 both occur twice and every other number occurs once: $M_1 = 98$ and $M_2 = 87$. The 11 numbers in the table are in decreasing order, so the median is the middle one: $Q_2 = 96$. The lower quartile is $Q_1 = 88$ and the upper quartile is $Q_3 = 99$. With the smallest value $Q_0 = 87$ and the largest value $Q_4 = 112$, the box plot is



Remark: $Q_4 = 112$ *is not an outlier, because with the interquartile range* $IQR = Q_3 - Q_1 = 11$, $112 < Q_3 + 1.5 \cdot IQR = 99 + 1.5 \cdot 11 = 115.5$.

c) Since p is the smallest of the 12 numbers, $p \le 87$. The median of the new data set is the mean of the two middle numbers, 96 and 93: $Q_2' = 94.5$. The new lower quartile is the median of the numbers below Q_2', so $Q_1' = \frac{88+87}{2} = 87.5$, and the new upper quartile is the median of the numbers above Q_2', so $Q_3' = \frac{99+98}{2} = 98.5$. Then $Q_3' - Q_1' = 11$, and the condition becomes $6.5 \cdot 11 < p < 8.5 \cdot 11$, that is, $71.5 < p < 93.5$. The primes between 72 and 94 are 73, 79,

83, and 89. 89 is not possible, because p must be the smallest of the 12 numbers. So Ludovic Fabregas scored 73, 79, or 83 goals.

Remark: In reality, Fabregas scored 79 goals in this period.

3. a) Which integers do not satisfy the following inequality?

$$\log_2 \frac{(12x+4)(4x-3)}{15x-4} - \log_{0.5} \frac{15x-4}{4x-3} \leq \log_2 (30x-50)$$

(5 points)

The sides of a triangle are $4n-3$, $15n-4$, and $12n+4$, where n is a positive integer.

b) For which values of n is the triangle right, acute, or obtuse? *(8 points)*

Solution: a) Because of the logarithm on the right side, $30x-50>0$, so $x>\frac{5}{3}$. Then $12x+4>0$, $15x-4>0$, and $4x-3>0$ also hold, so every term on the left side is defined. Since $\log_{0.5} u = -\log_2 u$, the inequality can be written as

$$\log_2 \frac{(12x+4)(4x-3)}{15x-4} + \log_2 \frac{15x-4}{4x-3} \leq \log_2 (30x-50)$$

By the product rule of logarithms,

$$\log_2 \left(\frac{(12x+4)(4x-3)}{15x-4} \cdot \frac{15x-4}{4x-3} \right) \leq \log_2 (30x-50), \qquad \text{that is,} \qquad \log_2 (12x+4) \leq \log_2 (30x-50)$$

The function $\log_2 x$ is strictly increasing, so this holds exactly when $12x+4 \leq 30x-50$, that is, $x \geq 3$. So the inequality does not hold for the integers $x \leq 2$.

b) Let $a=4n-3$, $b=15n-4$, and $c=12n+4$. Clearly $a<b$ for every positive integer n. $b<c$ means $15n-4<12n+4$, that is, $n<\frac{8}{3}$, so $n=1$ or $n=2$. If $n=1$, then $a=1$, $b=11$, $c=16$; these numbers violate the triangle inequality, so there is no such triangle.

If $n=2$, then $a=5$, $b=26$, $c=28$; these satisfy the triangle inequality, so this triangle exists. A triangle is acute, right, or obtuse exactly when the sum of the squares of its two shorter sides is greater than, equal to, or less than the square of the longest side. Since $5^2+26^2=701<28^2=784$, this triangle is obtuse.

If $n \geq 3$, then $4n-3<12n+4<15n-4$, and the triangle always exists, since $(4n-3)+(12n+4)>15n-4$ means $n+5>0$. We examine the inequality $(4n-3)^2+(12n+4)^2 \geq (15n-4)^2$. Expanding and rearranging, we get $-65n^2+192n+9 \geq 0$. The solutions of $65n^2-192n-9=0$ are $n_1=-\frac{3}{65}$ and $n_2=3$. For $n=3$ the sides are 9, 41, and 40: this is a right triangle with legs 9 and 40 and hypotenuse 41. For $n>3$, $-65n^2+192n+9<0$, so

$$(4n-3)^2+(12n+4)^2<(15n-4)^2$$

and the triangle is obtuse.

In summary: for $n=2$ and for $n>3$ the triangle exists and is obtuse, for $n=3$ it is a right triangle, and there is no n for which the triangle is acute.

4. A box contains 10 red balls, 13 green balls, and some blue balls. We pick a ball from the box at random.

   a) What is the probability that the ball is not red, if we know that the number of blue balls is to the number of non-blue balls as the number of green balls is to the number of non-green balls? *(3 points)*

   50 students reached the final round of a math competition. The jury awards 1 gold medal, 2 silver medals, and 3 bronze medals; every student can receive at most one medal.

   b) In how many ways can the 6 medals be awarded? *(3 points)*

   From the 44 students who did not receive a medal, the jury selects 20 whose good results it rates as equal, and a lottery decides which 6 of them receive a special prize. The names of the 20 students are written on slips of paper, and the slips are put into a box. 6 slips are drawn at random, without replacement.

   c) Of the 20 students in the lottery, 8 came from school A, 7 from school B, and 5 from school C. What is the probability that at least 2 of the 6 names drawn are students of school A? *(8 points)*

   Solution: a) Let the number of blue balls be k, a positive integer. By the condition,

$$\frac{k}{23} = \frac{13}{10+k}$$

   which leads to the quadratic equation $k^2 + 10k - 299 = 0$, with solutions $k_1 = 13$ and $k_2 = -23$. Only $k = 13$ is possible, so there are 13 blue balls and 36 balls in all. The probability that the ball is not red is

$$P\,(\text{not red}) = \frac{26}{36} = \frac{13}{18}$$

   b) The gold medal can be awarded in $\binom{50}{1} = 50$ ways. After that, the two silver medalists can be chosen in $\binom{49}{2} = 1176$ ways, and then the bronze medalists in $\binom{47}{3} = 16\,215$ ways. So the 6 medals can be awarded in $50 \cdot 1176 \cdot 16\,215 = 953\,442\,000$ ways.

   Remark: In practice, of course, the 6 best students receive the medals, and that has nothing to do with the number of possible selections.

   c) We compute the probability of the complement: that exactly 0 or exactly 1 of the 6 slips drawn has the name of a student of school A. There are $\binom{20}{6} = 38\,760$ ways to draw 6 slips. The number of draws with no student of school A is $\binom{8}{0}\binom{12}{6} = 924$, and with exactly one student of school A it is $\binom{8}{1}\binom{12}{5} = 6336$. So

$$P\,(\text{at least 2}) = 1 - \frac{924 + 6336}{38\,760} = 1 - \frac{7260}{38\,760} = \frac{525}{646} \approx 0.813$$

Part II

5. a) If we subtract p times a positive three-digit number from the square of the number (where p is a positive prime), we get exactly $20p^2$. Find the smallest and the largest such three-digit number. *(7 points)*

   G_1 and G_2 are trees with no common vertex. G_1 has $7q$ vertices and G_2 has $3r$ vertices, where q and r are positive primes. Connecting a vertex of G_1 to a vertex of G_2 by an edge, we get a tree G_3 with $q \cdot r$ edges.

   b) Find the number of vertices and edges of G_1 and of G_2, and the sum of the degrees in each of the two graphs. *(9 points)*

   Solution: a) Let the three-digit number be x. By the condition, $x^2 - px = 20p^2$, that is, $x^2 - px - 20p^2 = 0$. The discriminant of this quadratic equation in x is $81p^2 > 0$, and its solutions are

$$x_1 = \frac{p - 9p}{2} = -4p \qquad \text{and} \qquad x_2 = \frac{p + 9p}{2} = 5p$$

x_1 is negative, so it is not a solution. Since x has three digits, $99 < 5p \leq 999$, that is, $19.8 < p \leq 199.8$. The smallest prime in this interval is 23 and the largest one is 199. So the smallest such number is $5 \cdot 23 = 115$ and the largest one is $5 \cdot 199 = 995$.

b) A tree with n vertices has $n-1$ edges, so G_1 has $7q-1$ edges and G_2 has $3r-1$ edges. Connecting a vertex of G_1 to a vertex of G_2, we get a tree G_3 with $(7q-1)+(3r-1)+1$ edges. By the condition,

$$7q - 1 + 3r - 1 + 1 = qr$$

This can be written as $3r - 1 = q\,(r-7)$. Here $r \neq 7$, because then the right side would be 0 and the left side 20. So we can divide by $r-7$:

$$q = \frac{3r-1}{r-7} = \frac{3\,(r-7)+20}{r-7} = 3 + \frac{20}{r-7}$$

q is a positive integer, so $r-7$ must be a (positive or negative) divisor of 20:

$r-7$	1	-1	2	-2	4	-4	5	-5	10	-10	20	-20
r	8	6	9	5	11	3	12	2	17	-3	27	-13
q	23	-17	13	-7	8	-2	7	-1	5	1	4	2

Only $r = 17$, $q = 5$ satisfies all the conditions (both prime). So G_1 has $7q = 35$ vertices and 34 edges, and G_2 has $3r = 51$ vertices and 50 edges. The sum of the degrees is twice the number of edges: 68 in G_1 and 100 in G_2. (Check: G_3 has $35 + 51 = 86$ vertices and 85 edges, and indeed $85 = 5 \cdot 17$.)

6. We mark 28 points on a line in the plane. We also mark more than 7 points in the plane that are not on this line, so that no three of all the points are on a line, except for the 28 points on the line.

a) How many points did we mark outside the line, if the number of lines determined by all the marked points is 403? *(7 points)*

We mark n different points on a circle, where $n \geq 4$. Let A, B, and C denote the number of lines, triangles, and quadrilaterals determined by the n points.

b) Is it possible that $A + B = C$? (Justify your answer.) *(9 points)*

Solution: a) Let the number of points outside the line be k; then $k > 7$. Connecting each of the k points to each of the 28 points on the line gives $28k$ lines. To these we add the line containing the 28 points, and the lines through two of the k points, $\binom{k}{2} = \frac{k(k-1)}{2}$ of them. So

$$28k + 1 + \frac{k\,(k-1)}{2} = 403$$

which leads to the quadratic equation $k^2 + 55k - 804 = 0$, with solutions $k_1 = -67$ and $k_2 = 12$. So we marked 12 points outside the line. (Check: $28 \cdot 12 + 1 + 66 = 403$.)

b) Since the n points are on a circle, no three of them are on a line. So any two of the points determine a different line, any three of them a triangle, and any four of them a (convex) quadrilateral. So

$$A = \binom{n}{2} = \frac{n\,(n-1)}{2}, \qquad B = \binom{n}{3} = \frac{n\,(n-1)\,(n-2)}{6}, \qquad C = \binom{n}{4} = \frac{n\,(n-1)\,(n-2)\,(n-3)}{24}$$

and we look for the solutions of the equation

$$\frac{n\,(n-1)}{2} + \frac{n\,(n-1)\,(n-2)}{6} = \frac{n\,(n-1)\,(n-2)\,(n-3)}{24}$$

Since $n \geq 4$, we can divide both sides by $n\,(n-1)$:

$$\frac{1}{2} + \frac{n-2}{6} = \frac{(n-2)\,(n-3)}{24}$$

Multiplying by 24 and rearranging, we get $n^2 - 9n + 2 = 0$, with solutions $n_{1,2} = \frac{9 \pm \sqrt{73}}{2}$. Neither of them is an integer, so $A + B = C$ is not possible.

7. In the coordinate plane we mark the points $A\,(3,-4)$, $B\,(8,1)$, $C\,(6,5)$, $D\,(-1,-2)$, and $P\,(-2,1)$.

a) Prove that the five points are on a circle. *(4 points)*

b) Show that the quadrilateral $ABCD$ is a trapezoid, and compute its area. *(4 points)*

Consider the function $f:\mathbb{R}\to\mathbb{R}$, $f\,(x)=ax^2+bx+c$ whose graph passes through the points A, B, and C, and the function $g:\mathbb{R}\to\mathbb{R}$, $g\,(x)=dx^2+ex+f$ whose graph passes through the points A, C, and P.

c) Find the area of the region enclosed by the two graphs. *(8 points)*

Solution: a) We show that A and P are on the circumscribed circle of the triangle BCD. The midpoint of the segment BC is $F_1\,(7,3)$ and the midpoint of CD is $F_2\left(\frac{5}{2},\frac{3}{2}\right)$. $\overrightarrow{BC}=(-2,4)$, so $(-1,2)$ is a normal vector of the perpendicular bisector of BC; $\overrightarrow{CD}=(-7,-7)$, so $(1,1)$ is a normal vector of the perpendicular bisector of CD. The equations of the two perpendicular bisectors are $-x+2y=-1$ and $x+y=4$. The solution of this system is $x=3$, $y=1$, so the center of the circumscribed circle of BCD is $O\,(3,1)$, its radius is $R=OD=\sqrt{4^2+3^2}=5$, and its equation is

$$(x-3)^2+(y-1)^2=25$$

Substituting, we can check that the coordinates of both $A\,(3,-4)$ and $P\,(-2,1)$ satisfy this equation. So the five points are on a circle.

b) $\overrightarrow{AB}=(5,5)$ and $\overrightarrow{CD}=(-7,-7)$, so the lines AB and CD both have slope 1. So $ABCD$ is a trapezoid with parallel sides AB and CD (and since its vertices are on a circle, it is an isosceles trapezoid). The lengths of the parallel sides are

$$AB=\sqrt{5^2+5^2}=5\sqrt{2}\qquad\text{and}\qquad CD=\sqrt{(-7)^2+(-7)^2}=7\sqrt{2}$$

The equation of the line CD is $-x+y+1=0$. The height of the trapezoid is the distance of $A\,(3,-4)$ from this line:

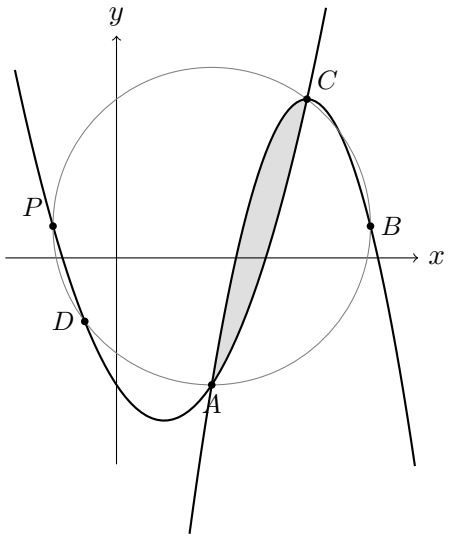
$$m=\frac{|-3-4+1|}{\sqrt{2}}=\frac{6}{\sqrt{2}}=3\sqrt{2}$$

So the area of the trapezoid is $T=\frac{5\sqrt{2}+7\sqrt{2}}{2}\cdot 3\sqrt{2}=36$.

c) Substituting the coordinates of A, B, and C into $y=ax^2+bx+c$, and those of A, C, and P into $y=dx^2+ex+f$, we get the systems

$$\begin{aligned} 9a+3b+c &= -4, \qquad 64a+8b+c=1, \qquad 36a+6b+c=5\\ 9d+3e+f &= -4, \qquad 36d+6e+f=5, \qquad 4d-2e+f=1 \end{aligned}$$

Their solutions are $a=-1$, $b=12$, $c=-31$ and $d=\frac{1}{2}$, $e=-\frac{3}{2}$, $f=-4$. The two parabolas $y=-x^2+12x-31$ and $y=\frac{1}{2}x^2-\frac{3}{2}x-4$ meet at A and C, as the figure shows. (The point D is also on the second parabola, but we do not need this.)



The area of the region between the two parabolas is

$$\begin{aligned} T &= \int_3^6 (f(x) - g(x))\,dx = \int_3^6 \left(-\frac{3}{2}x^2 + \frac{27}{2}x - 27\right) dx = \left[-\frac{1}{2}x^3 + \frac{27}{4}x^2 - 27x\right]_3^6 \\ &= (-108 + 243 - 162) - \left(-\frac{27}{2} + \frac{243}{4} - 81\right) = -27 - \left(-\frac{135}{4}\right) = \frac{27}{4} = 6.75 \end{aligned}$$

8. a) Find the natural numbers n for which the inequality $n! > 3^n$ holds, and also those for which it does not hold. *(5 points)*

Let $n > 1$ be a positive integer, and define the sequences a_n and b_n as follows:

$$a_n = \binom{n}{2} + \binom{n}{3}, \qquad b_n = \binom{2n}{3}$$

b) Prove that the sequence $c_n = \frac{a_n}{b_n}$ is strictly decreasing. *(7 points)*

The first term of the geometric sequence d_n is $d_1 = 24$ and its common ratio is $q_d = \frac{6}{22}$; the first term of the geometric sequence e_n is $e_1 = 21$ and its common ratio is $q_e = \frac{15}{22}$. Let S_d and S_e denote the infinite sums $d_1 + d_2 + d_3 + \cdots$ and $e_1 + e_2 + e_3 + \cdots$.

c) Find the exact value of $S_d + S_e$. *(4 points)*

Solution: a) With a calculator we find that $n! > 3^n$ does not hold for the natural numbers $n \le 6$: for $n = 0$, $n! = 3^n = 1$, and for $1 \le n \le 6$, $n! < 3^n$ (for example, $6! = 720 < 3^6 = 729$). For $n = 7$, $7! = 5040 > 3^7 = 2187$. We prove by induction that $n! > 3^n$ for every natural number $n \ge 7$. We have just seen that it is true for $n = 7$. Suppose that $k! > 3^k$ for some $k \ge 7$. Multiplying both sides by the positive number $k + 1$, we get $(k+1)! > 3^k(k+1)$, and since $k + 1 > 3$, $(k+1)! > 3^k(k+1) > 3^{k+1}$. So the inequality also holds for $k + 1$.

In summary, $n! > 3^n$ does not hold for $n \in \{0, 1, 2, 3, 4, 5, 6\}$, and it holds for every natural number $n \ge 7$.

Note: KoMaL's solution writes $k > 7$ in the induction step; $k \ge 7$ is what is needed (and $k + 1 > 3$ holds for it).

b) By the identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$,

$$a_n = \binom{n}{2} + \binom{n}{3} = \binom{n+1}{3} = \frac{(n+1)\,n\,(n-1)}{6}$$

and

$$b_n = \binom{2n}{3} = \frac{2n\,(2n-1)\,(2n-2)}{6} = \frac{4n\,(n-1)\,(2n-1)}{6}$$

So for $n > 1$,

$$c_n = \frac{a_n}{b_n} = \frac{n+1}{4\,(2n-1)} = \frac{n+1}{8n-4}$$

and

$$c_{n+1} - c_n = \frac{n+2}{8n+4} - \frac{n+1}{8n-4} = \frac{(n+2)\,(8n-4) - (n+1)\,(8n+4)}{(8n+4)\,(8n-4)} = \frac{-12}{(8n+4)\,(8n-4)}$$

For $n > 1$ the denominator is positive, so $c_{n+1} - c_n < 0$, that is, $c_{n+1} < c_n$ for every n. So the sequence c_n is strictly decreasing.

c) In both geometric sequences the common ratio is between 0 and 1, so both infinite sums exist. By the formula for the sum of an infinite geometric series,

$$S_d = \frac{24}{1 - \frac{6}{22}} = 33 \qquad \text{and} \qquad S_e = \frac{21}{1 - \frac{15}{22}} = 66$$

so $S_d + S_e = 99$.

9. The legs of the right triangle ABC are $BC = a$ and $CA = b$, its hypotenuse is $AB = c$, and $\angle BAC = 15°$. We rotate the right triangle about the line of the leg BC, and also about the line of the leg CA.

a) Find the exact value of the ratio of the volumes of the two solids we get. *(6 points)*

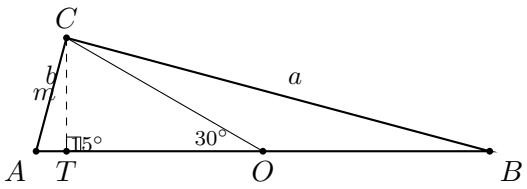
In the acute triangle ABC, $BC = a$, $CA = b$, $AB = c$, and $a < b < c$. We rotate the triangle about the line of the side a, then about the line of the side b, and finally about the line of the side c. The volumes of the three solids, in this order, are three consecutive terms of an arithmetic sequence.

b) Prove that the length of the side b is the harmonic mean of the lengths of the sides a and c. *(10 points)*

Solution: a) Rotating about BC, we get a cone with base radius b and height a; rotating about CA, we get a cone with base radius a and height b. Their volumes are

$$V_a = \frac{b^2\pi a}{3} \qquad \text{and} \qquad V_b = \frac{a^2\pi b}{3}$$

so $\frac{V_a}{V_b} = \frac{b}{a}$. To find the exact value of $\frac{b}{a}$, let O be the center of the circumscribed circle of the triangle ABC, m the height from C, and T its foot.



The circumradius is $OA = OB = OC = \frac{c}{2}$ (by Thales' theorem, O is the midpoint of AB). The triangle OCA is isosceles, so $\angle OCA = \angle CAO = 15°$, and by the exterior angle theorem $\angle COT = 30°$. So the right triangle COT is a semi-regular (30°-60°-90°) triangle, and $\frac{c}{2} = 2m$, that is,

$$m = \frac{c}{4}$$

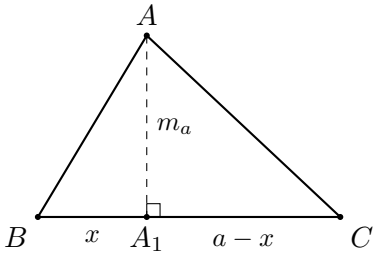
Twice the area of the triangle ABC is $ab = cm$, so $ab = \frac{c^2}{4}$, and by the Pythagorean theorem,

$$a^2 + b^2 = 4ab$$

Dividing both sides by a^2 and substituting $x = \frac{b}{a}$, we get $x^2 - 4x + 1 = 0$, with solutions $x_1 = 2 - \sqrt{3}$ and $x_2 = 2 + \sqrt{3}$. Since the angle at A is 15°, the angle at B is 75°, so $b > a$, that is, $x > 1$, while $x_1 < 1$. So $\frac{b}{a} = 2 + \sqrt{3}$, and

$$\frac{V_a}{V_b} = \frac{b}{a} = 2 + \sqrt{3}$$

b) The triangle is acute, so all three heights are inside the triangle. Let m_a, m_b, and m_c be the heights to the sides a, b, and c, and let A_1 be the foot of the height from A.



Rotating the triangle about BC, we get two cones with a common base, a circle of radius m_a. With $BA_1 = x$ and $A_1C = a - x$, the total volume is

$$V_a = \frac{m_a^2 \pi x}{3} + \frac{m_a^2 \pi (a - x)}{3} = \frac{m_a^2 \pi a}{3}$$

In the same way, rotating about the sides b and c,

$$V_b = \frac{m_b^2 \pi b}{3} \qquad \text{and} \qquad V_c = \frac{m_c^2 \pi c}{3}$$

By the condition, V_a, V_b, V_c are consecutive terms of an arithmetic sequence, so $2V_b = V_a + V_c$, that is,

$$2m_b^2 b = m_a^2 a + m_c^2 c$$

Twice the area T of the triangle is $2T = am_a = bm_b = cm_c$, so the equation can be written as $2 \cdot 2T \cdot m_b = 2T \cdot m_a + 2T \cdot m_c$, that is, $2m_b = m_a + m_c$. Substituting $m_a = \frac{2T}{a}$, $m_b = \frac{2T}{b}$, and $m_c = \frac{2T}{c}$ and simplifying,

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}, \qquad \text{so} \qquad b = \frac{2}{\frac{1}{a} + \frac{1}{c}}$$

So the length of b is indeed the harmonic mean of the lengths of a and c.

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