author: Péter Polák (Budapest)
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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. a) Three consecutive terms of an arithmetic sequence (in this order) are y, x, and $4x-5$. Find these numbers, if their sum is 1500. *(4 points)*

   b) How many geometric sequences are there whose first 5 terms include 2, 8, and 32, if the order of the terms matters? *(5 points)*

   c) Put the cardinalities of the following sets in increasing order. Justify your answer.

   $A = \{\text{the rational solutions of the equation } x^2+2x+1=2\}$

   $B = \{\text{the positive divisors of } 40\}$

   $C = \left\{n : 3^{-n+1} > \frac{1}{27^3},\, n \in \mathbb{N}\right\}$ *(5 points)*

   Solution: a) The sum of the three numbers is three times the middle term: $3x = 1500$, so $x = 500$. The common difference of the sequence is $(4x-5)-x = 3x-5 = 1495$. So the three numbers are -995, 500, and 1995; they indeed form an arithmetic sequence.

   b) The given numbers themselves form a geometric sequence, with ratio $q = 4$ or $q = \frac{1}{4}$. So there are two possibilities: (I) they are neighbors among the five terms too, or (II) they are the first, third, and fifth terms.

   (I) If they are neighboring terms, then the first of them can be the first, second, or third term: three possibilities. In each of them there are two sequences ($q = 4$ or $q = \frac{1}{4}$), so there are six such sequences.

   (II) If they are not neighbors, then $q^2 = 4$ or $q^2 = \frac{1}{4}$. In both cases there are two possible ratios: $q = \pm 2$ or $q = \pm\frac{1}{2}$, so there are 4 such sequences.

   So there are 10 such geometric sequences in all.

   c) $x^2+2x+1 = (x+1)^2 = 2$ has solutions $-1 \pm \sqrt{2}$, which are irrational, so $|A| = 0$. $40 = 2^3 \cdot 5$, so $|B| = 4 \cdot 2 = 8$. Since $\frac{1}{27^3} = 3^{-9}$ and the base 3 exponential function is strictly increasing, $3^{-n+1} > 3^{-9}$ holds exactly when $-n+1 > -9$, that is, $n < 10$. So $C = \{0, 1, \ldots, 9\}$ and $|C| = 10$. The order of the cardinalities is $|A| < |B| < |C|$.

2. a) Prove the following statement:

"If the lengths of all three sides of a right triangle are positive integers, then the length of the height to the hypotenuse is rational." *(4 points)*

b) State the converse of the statement in part a), and decide whether it is true or false. Justify your answer. *(3 points)*

c) We added the squares of 27 different primes, and got 155 787. Could 3 be among the primes? *(5 points)*

Solution: a) Let the legs be a and b, the hypotenuse c, and the height to the hypotenuse m. Then the area is $T = \frac{ab}{2} = \frac{cm}{2}$, so $m = \frac{ab}{c}$: m is a quotient of integers, so it is rational.

b) The converse: "If the height to the hypotenuse of a right triangle is rational, then the lengths of all three sides are integers." This is false. Consider a square whose diagonals are 10 units long. In the right triangle that is half of this square, $c = 10$ and $m = 5$, but $a = b = \frac{10}{\sqrt{2}} = 5\sqrt{2}$, which are irrational.

c) If a number is not divisible by 3, then it can be written as $3k \pm 1$, where k is an integer, and $(3k \pm 1)^2 = 9k^2 \pm 6k + 1$. So its square always leaves remainder 1 when divided by 3. Suppose that 3 is among the 27 primes. Then the sum consists of 26 numbers with remainder 1 and one number with remainder 0, so the sum leaves remainder 2 when divided by 3. But $3 \mid 155\,787$ (its digit sum is 33), so this is impossible. Even if there are 27 such primes, 3 is certainly not among them.

3. 90 students took part in a camp, and they were housed in wooden cabins. Each cabin had four, eight, or nine beds, and there were exactly five cabins with eight beds. After the students moved into the cabins, there were no empty beds.

a) On arrival, the host gave the keys to one of the accompanying teachers one by one: first the keys of the four-bed cabins, then those of the eight-bed cabins, and finally those of the nine-bed cabins. In how many different orders could the host do this, if every cabin has one key and the keys are all different? *(4 points)*

One afternoon the students could choose from several different hikes. The 90 students formed 6 groups in all. We know the following about the sizes of the six groups:

- the size of every group is divisible by six;
- there is a group with exactly half as many students as another group;
- the sizes have two modes, and one of them is equal to the median;
- the range of the sizes is 30.

b) How many students were in each group? *(6 points)*

Solution: a) Let the number of four-bed cabins be a and the number of nine-bed cabins b, positive integers. Then $4a + 9b + 40 = 90$, that is, $4a + 9b = 50$. So $b < 6$, and $9b$ must leave remainder 2 when divided by 4, so the only possibility is $b = 2$, and then $a = 8$. So the keys could be handed over in

$$8! \cdot 5! \cdot 2! = 9\,676\,800$$

different orders.

b) By the first and the fourth conditions, the sizes of the six groups can be written as $6a$, $6b$, $6c$, $6d$, $6e$, $6a + 30$, where $0 < 6a \le 6b \le 6c \le 6d \le 6e \le 6a{+}30$ and a, b, c, d, e are positive integers. Then $6a{+}6b{+}6c{+}6d{+}6e{+}6a{+}30 = 90$, that is, $2a + b + c + d + e = 10$.

Here b is at most 2, since otherwise $b + c + d + e \ge 12 > 10$. If $b = 2$, then the only possibility is $a = 1$, $b = c = d = e = 2$, but then the data have only one mode. So $b = 1$, and then $a = 1$ too. So one of the modes is 1 (that is, 6 students), and $c + d + e = 7$.

The median is the mean of the two middle numbers, $\frac{c+d}{2}$, and since it is also a mode, it is one of the data values; this is only possible if $c = d$. But $c = d = 1$ is not possible, since then there would be only one mode. So $c = d = 2$, and then $e = 3$.

So a, b, c, d, e, $a + 5$ are 1, 1, 2, 2, 3, 6. We also check that one of the numbers is twice another one, and this indeed holds (6 and 12, or 18 and 36). So the group sizes are 6, 6, 12, 12, 18, and 36.

4. a) Consider the following points in the coordinate plane:

$$A\left(\log_3 r, \log_3 5r\right), \qquad B\left(-\log_3 \frac{r}{3}, \log_{\frac{1}{3}} 10\right), \qquad C\left(\log_r 1, \log_r 1\right)$$

Find the value of $r \in \mathbb{R}^+ \setminus \{1\}$, if the centroid of the triangle ABC is the point $S\left(\frac{1}{3}, \frac{1}{3}\right)$. *(5 points)*

b) Consider the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x^3 + 6x - 6x^2 - 2$. Give the rule of the function $g(x)$, defined on the real numbers, for which $g(f(x)) = x$. *(4 points)*

c) Find the value of the nonzero real parameter p in the rule of the function $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2 + px + p$, so that the graphs of g and of the linear function $x \mapsto x + 1$ have exactly one intersection point. *(6 points)*

Solution: a) The coordinates of the centroid are the averages of the corresponding coordinates of the vertices. For every $r > 0$, $r \neq 1$, $\log_r 1 = 0$, so multiplying by three, we get the equations

$$\log_3 r - \log_3 \frac{r}{3} = 1 \qquad \text{and} \qquad \log_3 5r + \log_{\frac{1}{3}} 10 = 1$$

By the rules of logarithms, the first equation is $\log_3 \frac{r}{r/3} = \log_3 3 = 1$, so every r in the domain is a solution. In the second equation we change to base 3:

$$\log_3 5r + \frac{\log_3 10}{\log_3 \frac{1}{3}} = \log_3 5r - \log_3 10 = \log_3 \frac{5r}{10} = \log_3 \frac{r}{2} = \log_3 3$$

The logarithm function is one-to-one, so $\frac{r}{2} = 3$, and $r = 6$. This value satisfies the conditions.

b) $f(x) = 2\left(x^3 - 3x^2 + 3x - 1\right) = 2(x-1)^3$. So $g(x) = \sqrt[3]{\frac{x}{2}} + 1$. Indeed, $g(f(x)) = \sqrt[3]{\frac{2(x-1)^3}{2}} + 1 = x$.

c) The graphs meet where the function values are equal: $x^2 + px + p = x + 1$, that is, $x^2 + (p-1)x + (p-1) = 0$. This quadratic equation has exactly one solution if its discriminant is zero:

$$D = (p-1)^2 - 4(p-1) = (p-1)(p-5) = 0$$

So $p = 1$ or $p = 5$, and both values indeed satisfy the conditions.

Part II

5. a) The base edges and the lateral edges of a right prism with a regular hexagon base are equally long. If we divide the numerical value of its volume by the numerical value of its surface area (in the same system of units), we get 1.392. How many units long is an edge of the prism? Round your answer to a whole number. *(9 points)*

b) We choose three edges of a prism with a hexagon base. What is the probability that exactly two of the three edges have a common point? *(7 points)*

Solution: a) Let an edge of the prism be a. The base consists of six equilateral triangles, so

$$V = 6 \cdot \frac{a^2\sqrt{3}}{4} \cdot a = \frac{3\sqrt{3}}{2}a^3 \qquad \text{and} \qquad A = 2 \cdot 6 \cdot \frac{a^2\sqrt{3}}{4} + 6a^2 = 3a^2\left(\sqrt{3} + 2\right)$$

After simplifying, the quotient is

$$\frac{V}{A} = a \cdot \frac{\sqrt{3}}{2\left(\sqrt{3}+2\right)} = 1.392$$

so $a \approx 6$ units.

b) The prism has 18 edges and 12 vertices, and exactly 3 edges meet at every vertex. Three of the 18 edges can be chosen in $\binom{18}{3} = 816$ ways. For a favorable choice, we choose the common vertex and two of the three edges meeting there: $12 \cdot \binom{3}{2} = 36$ ways. The third edge cannot have a common point with these two: the 3 edges at the common vertex and the $2 + 2$ other edges at the other two endpoints are excluded, so only $18 - 3 - 2 - 2 = 11$ edges can be chosen. So

$$P = \frac{12 \cdot \binom{3}{2} \cdot 11}{\binom{18}{3}} = \frac{396}{816} = \frac{33}{68} \approx 0.4853$$

6. A marketing company is making a survey of 100 people about a new plant-based chocolate. First the participants decide, without knowing the product, only based on its name, whether they would buy it. Then they taste the product and are asked again whether they would buy it. The answers to the two questions are in the table below.

	based on the name	after tasting
yes	45	72
no	55	28

a) At most how many people could have changed their minds about buying the product because of the tasting? *(4 points)*

b) Is it possible that exactly 42 people did not change their minds because of the tasting? *(4 points)*

The same company organizes the advertising campaign of a board game. One of the accessories of the game is a regular icosahedron. It has 20 faces, and when it is thrown, it lands on each of them with the same probability. The faces show only the numbers 4, 7, and 11 (exactly one number on each face). The rule book says at one place: "two of the three numbers are thrown with the same probability, and the expected value of a throw is 7.35".

c) How many faces show a 4? *(8 points)*

Solution: a) At most 28 people could change from "yes" to "no" (that many said "no" after tasting), and at most 55 from "no" to "yes". So at most $28 + 55 = 83$ people changed their minds, and this can indeed happen (if all 55 "no" answers became "yes" and 28 of the 45 "yes" answers became "no"). So at most 83 people changed their minds.

b) Suppose that k people changed from "yes" to "no"; then $100 - 42 - k = 58 - k$ people changed from "no" to "yes". So the number of "yes" answers after tasting is

$$45 - k + 58 - k = 103 - 2k = 72$$

but then $k = 15.5$ is not an integer, a contradiction. So this is not possible.

c) Let the two equal probabilities be p; then the probability of the third number is $1 - 2p$. The numbers appear on $20p$, $20p$, and $20\,(1 - 2p)$ faces.

Case I: $4p + 7p + 11\,(1 - 2p) = 7.35$ gives $p = \frac{73}{220}$. This is not a solution, since $20p$ is not an integer.

Case II: $11p + 7p + 4\,(1 - 2p) = 7.35$ gives $p = \frac{67}{200}$. This is not a solution either, since $20p$ is not an integer.

Case III: $4p + 11p + 7\,(1 - 2p) = 7.35$ gives $p = \frac{7}{20}$. So there are seven 4s and seven 11s, and six 7s on the faces.

So 7 faces show a 4.

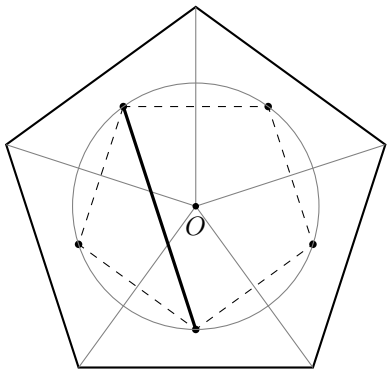
Note: KoMaL's solution gives $p = \frac{63}{200}$ in case II; it is $\frac{67}{200}$, which does not change the conclusion. In the original problem the product is simply a new chocolate; we made it plant-based.

7. One floor of a lookout tower in the forest has the shape of a regular pentagon with 5 meter sides. Two different "photo points" were marked on the floor, where you can stand to take the best photos of the landscape. Each of the two points is equally far from the center of the circumscribed circle of the pentagon and from two neighboring vertices.

a) At most how far apart can the two "photo points" be? *(11 points)*

b) The lookout gives out a free booklet about the plants of the forest. It contains a figure with pictures of ten plants typical of the area; two pictures are connected by a line if the two plants often grow next to each other. Balambér counted that there are two pictures from which only 3 lines start, and from each of the others at least six. His sibling says that this means that you can get from any picture to any other along the lines. Is the sibling right? *(5 points)*

Solution: a) A photo point is the center of the circumscribed circle of a triangle formed by the center O of the pentagon and two neighboring vertices. There are 5 such points, and by the rotational symmetry they form a regular pentagon. In a regular pentagon all sides are equal, all diagonals are equal, and the diagonals are longer than the sides. So we are looking for the length of a diagonal of this small pentagon.



The triangle formed by O and two neighboring vertices is isosceles, with angle $\frac{360^\circ}{5} = 72^\circ$ at O and opposite side 5 m. So by the law of sines, its circumradius is $\rho = \frac{5}{2\sin 72^\circ}$, and the photo points are on the circle of radius ρ around O. A diagonal of the small regular pentagon is a chord of this circle with central angle 144°, so its length is

$$2\rho \sin 72^\circ = 2 \cdot \frac{5}{2\sin 72^\circ} \cdot \sin 72^\circ = 5 \text{ m}$$

So the two photo points can be at most 5 meters apart.

Remark: KoMaL's solution shows this with congruent rhombuses and isosceles triangles: the distance in question turns out to be equal to a side of the original pentagon.

b) The figure is a simple graph with 10 vertices, in which two vertices have degree 3 and the others have degree at least 6. The question is whether this graph is connected. Suppose that it is not connected, so it has at least two components. A component that contains a vertex of degree at least 6 has at least 7 vertices, so every other component has at most 3 vertices. But such a component contains a vertex of degree 3 or at least 6, so it must have at least 4 vertices. This is a contradiction, so the graph is connected: you can get from any picture to any other along the lines, and the sibling is right.

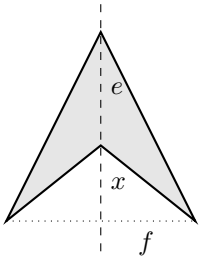
Note: In the original problem the figure is a food chain of ten animals; we changed it to plants that grow together. The graph is the same.

8. a) Rotating a concave kite about its axis of symmetry, we get a solid of volume $\frac{500\pi}{3}$ dm^3. Find the area of the kite, if one of its diagonals is 5 dm long. *(7 points)*

b) We found the zeros of the function $f(x) = \left(x^2-1\right)\left(x^2-4\right)$ defined on the real numbers. Then we computed the definite integral of f between the second and third largest zeros, and got a number that is not an integer. What is the smallest positive number we can multiply f by so that this definite integral becomes an integer? *(5 points)*

c) Write the equation of a tangent line of the function $f(x) = x^4 - 5x^2 + 4$ that is parallel to the x-axis. *(4 points)*

Solution: a) Rotating the concave kite, we get the solid between two right circular cones with the same base. Let e be the diagonal on the axis of symmetry, f the other diagonal (which lies outside the kite), and x the distance of the concave vertex from the line of f.



Three times the volume is

$$3V = \left(\frac{f}{2}\right)^2 \pi\,(e+x) - \left(\frac{f}{2}\right)^2 \pi x = \left(\frac{f}{2}\right)^2 \pi e = 500\pi$$

There are two cases, depending on which diagonal is 5 dm long.

Case I: $f = 5$: $\left(\frac{5}{2}\right)^2 e = 500$, so $e = 80$ dm, and the area is $T = \frac{5\cdot 80}{2} = 200$ dm^2.

Case II: $e = 5$: $\left(\frac{f}{2}\right)^2 \cdot 5 = 500$, so $f = 20$ dm, and the area is $T = \frac{5\cdot 20}{2} = 50$ dm^2.

b) A product is zero exactly when one of its factors is zero, so the zeros of f are ± 1 and ± 2. The second largest of these is 1 and the third largest is -1. Multiplying out, $f(x) = x^4 - 5x^2 + 4$, and

$$\int_{-1}^{1} \left(x^4 - 5x^2 + 4\right) dx = \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{-1}^{1} = \frac{76}{15}$$

Since $\int af(x)\,dx = a\int f(x)\,dx$, multiplying f by a multiplies the integral by a. The smallest positive integer a for which $\frac{76a}{15}$ is an integer is 15. (If a does not have to be an integer, the smallest positive a that makes the integral a (positive) integer is $\frac{15}{76}$, which gives 1.)

Note: KoMaL's answer is 15.

c) $f'(x) = 4x^3 - 10x$. The slope of the tangent line is 0, so $4x^3 - 10x = 2x\left(2x^2 - 5\right) = 0$, that is, $x_1 = 0$, $x_{2,3} = \pm\frac{\sqrt{10}}{2}$. So there are three such points. If $x = 0$, then $f(0) = 4$, and the tangent line is $y = 4$. At the other two points, $x^2 = \frac{5}{2}$ and $f(x) = \frac{25}{4} - \frac{25}{2} + 4 = -\frac{9}{4}$, so the tangent line there is $y = -\frac{9}{4}$.

Note: KoMaL's solution gives $y = \frac{5}{4}$ for the second tangent line; the correct value is $-\frac{9}{4}$.

9. a) Prove that for any $x, y \in \left[\frac{\pi}{7}, \frac{3\pi}{7}\right]$, the dot product of the vectors $\mathbf{a}(\tan x, \tan y)$ and $\mathbf{b}\left(\frac{\cos x}{\sin x}, \frac{\cos y}{\sin y}\right)$ is 2. *(4 points)*

b) Solve the following equation over the set of pairs of positive real numbers.

$$\frac{x}{y^2} + \frac{1}{xy} = \frac{4-x}{y} - \frac{1}{x}$$

(5 points)

c) We know that the following infinite sum is convergent and its value is a negative real number. Find the possible values of $p \in \mathbb{R}$.

$$\frac{1}{p+1} + \frac{p^2-1}{p+1} + \frac{\left(p^2-1\right)^2}{p+1} + \frac{\left(p^2-1\right)^3}{p+1} + \cdots < 0$$

(7 points)

Solution: a) All the expressions are defined at every point of the given interval. The coordinates of $\mathbf{b}$ are $\cot x$ and $\cot y$, the reciprocals of the corresponding coordinates of $\mathbf{a}$. So

$$\mathbf{a}\cdot\mathbf{b} = \tan x \cdot \frac{1}{\tan x} + \tan y \cdot \frac{1}{\tan y} = 1 + 1 = 2$$

b) Multiplying the equation by the positive number y and rearranging, we get

$$\frac{x}{y} + \frac{y}{x} + x + \frac{1}{x} = 4$$

The sum of a positive number and its reciprocal is at least 2, with equality only if the number is 1. So the left side is at least 4, and equality must hold in both pairs: $\frac{x}{y} = 1$ and $x = 1$. So the only solution is $x = y = 1$.

c) The sum is an infinite geometric series with first term $\frac{1}{p+1}$ and ratio $p^2 - 1$. It is defined if $p \neq -1$. An infinite geometric series converges exactly when its ratio is between -1 and 1: $-1 < p^2 - 1 < 1$, so $0 < p^2 < 2$, that is, $-\sqrt{2} < p < \sqrt{2}$ and $p \neq 0$. Then the sum is

$$\frac{\frac{1}{p+1}}{1-(p^2-1)} = \frac{1}{(p+1)(2-p^2)} < 0$$

This fraction is negative exactly when $p+1$ and $2-p^2$ have opposite signs. Since $p^2 < 2$, $2-p^2 > 0$, so $p+1 < 0$, that is, $p < -1$. So the inequality holds exactly when $-\sqrt{2} < p < -1$.

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