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Solutions: KoMaL website

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the largest subset of the set of natural numbers that can be the domain of the following expressions.

   a) $\log_x\left(-2x^2-7x+15\right)$ *(6 points)*

   b) $\sqrt{\frac{x^2-2x}{-2x^2-7x+15}}$ *(6 points)*

   Solution: a) Because of the base, $x>0$ and $x\neq 1$; because of the argument, $-2x^2-7x+15>0$. The roots of $-2x^2-7x+15$ are -5 and 1.5, so the inequality holds for $-5<x<1.5$. The only natural numbers here are 0 and 1, and neither of them can be the base. So the expression is not defined for any natural number: the set in question is the empty set.

   b) By the definition of the square root, $\frac{x^2-2x}{-2x^2-7x+15}\geq 0$. The numerator $x(x-2)$ is positive if $x<0$ or $x>2$, negative if $0<x<2$, and zero if $x=0$ or $x=2$. The denominator is positive if $-5<x<1.5$, and negative if $x<-5$ or $x>1.5$. So the fraction is nonnegative for $-5<x\leq 0$ and for $1.5<x\leq 2$, and the expression is defined for the natural numbers 0 and 2.

2. In an online game, we have to move a ladybug figure from the origin of the coordinate system to the point $F(8,4)$. In each step, the ladybug can only move one unit to the right or up.

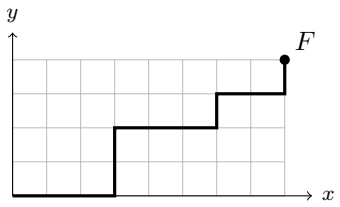
   a) In how many ways can the ladybug get from the origin to F? (Two routes are different if one of them steps on a lattice point that the other does not.) *(6 points)*

   b) The point $F(8,4)$ is the focus of a parabola. What can the equation of the parabola be, if its axis is parallel to the y-axis and it meets the y-axis at the point with second coordinate 10? *(6 points)*

   Solution: a) The ladybug gets from the origin to $F(8,4)$ in 8 steps to the right and 4 steps up. Writing r for right and u for up, a route is, for example, $rrrrrrrruuuu$ or $urururrrrrrr$. The different orders of the letters give the possible routes:

$$\frac{12!}{8!\cdot 4!}=495$$

   So the ladybug can get to F in 495 ways.



b) Let the vertex of the parabola be $T(u, v)$ and its parameter $p > 0$; the focus is at distance $\frac{p}{2}$ from the vertex. The parabola meets the y-axis at 10, so the point $P(0, 10)$ is on it. The axis is parallel to the y-axis, so there are two cases.

If the parabola opens downward, its vertex is $T\left(8, 4 + \frac{p}{2}\right)$, and its equation is $y - v = -\frac{(x-u)^2}{2p}$. Substituting the coordinates of T and P:

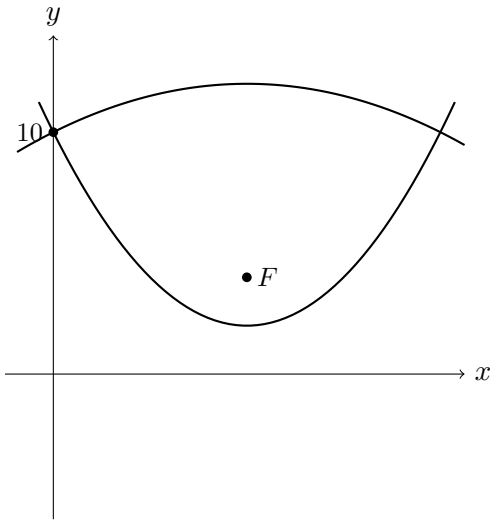
$$10 - \left(4 + \frac{p}{2}\right) = -\frac{(0-8)^2}{2p}$$

which leads to $p^2 - 12p - 64 = 0$, with positive solution $p = 16$. So one possible parabola is $y - 12 = -\frac{(x-8)^2}{32}$.

If the parabola opens upward, its vertex is $T\left(8, 4 - \frac{p}{2}\right)$, and its equation is $y - v = \frac{(x-u)^2}{2p}$. Substituting:

$$10 - \left(4 - \frac{p}{2}\right) = \frac{(0-8)^2}{2p}$$

which leads to $p^2 + 12p - 64 = 0$, with positive solution $p = 4$. So the other possible parabola is $y - 2 = \frac{(x-8)^2}{8}$.



3. Every member of an after-school group folds paper airplanes all afternoon. From experience, 3 percent of the folded airplanes are defective: they cannot be made to fly.

a) What is the probability that if we choose 5 of the paper airplanes, with replacement, there will be a defective one among them? *(5 points)*

A student made statistics of the number of paper airplanes finished per 10 minutes. The sum of the median and the lower quartile of the data is 28, the difference of the upper and lower quartiles is 6, and the difference of the upper quartile and the median is 4.

b) Find these values for the data. *(4 points)*

c) Draw a box plot of the data, if the minimum is 8, the maximum is 22, and there are no outliers. *(3 points)*

d) The members of the group decided to revive an old custom: they surprise each other with postcards during the summer break. So they exchanged their home addresses with each other (mutually). They imagined the exchanges as a graph, with the edges representing the address exchanges. If there had been two more of them (who also exchange addresses with everybody), the graph would have had 45 more edges. How many students were in the group? *(3 points)*

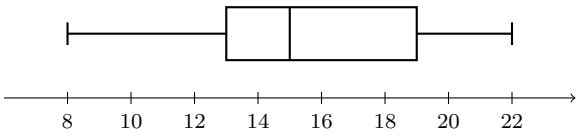
Solution: a) If there are defective airplanes among the chosen ones, there can be 1, 2, or more, even 5. So it is easier to compute the probability of the complement first: that there is no defective one among them.

$$P(\text{there is a defective one}) = 1 - P(\text{no defective one}) = 1 - 0.97^5 \approx 0.14$$

So the probability that there is a defective airplane among the chosen ones is about 14 percent.

b) Let the lower quartile be a, the median m, and the upper quartile f. Then $m + a = 28$, $f - a = 6$, and $f - m = 4$. The solution of this system is $a = 13$, $m = 15$, and $f = 19$.

c) (No outliers: $1.5 \cdot (19 - 13) = 9$, and $13 - 9 = 4 < 8$, $19 + 9 = 28 > 22$.)



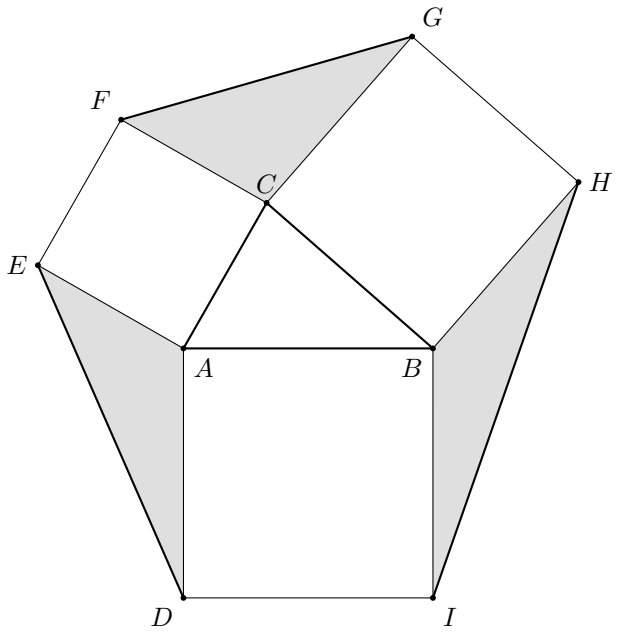
d) If n people exchange addresses, the graph with n vertices has $\binom{n}{2}$ edges. With two more people it has $\binom{n+2}{2}$ edges. So

$$\binom{n+2}{2} - \binom{n}{2} = \frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2} = 2n + 1 = 45$$

so $n = 22$: there were 22 students in the group.

Another way: the two new people would each exchange addresses with the n others and with each other, so $2n+1 = 45$, and $n = 22$.

4. On each side of a triangle we draw a square outward, with side equal to that side of the triangle. Connecting the free vertices of neighboring squares, we get the triangles ADE, CFG, and BHI, as in the figure.



a) Prove that the area of each of these triangles is equal to the area of the original triangle ABC. *(4 points)*

b) Find the area of the hexagon $DEFGHI$, if the sides of the original triangle are 6 cm, 7 cm, and 8 cm long. *(6 points)*

c) Give the truth values of the following statements, where α is an arbitrary real number.

A: $\cos(270^\circ + \alpha) = \sin\alpha$

B: $1 + \cos 2\alpha = 2\sin^2\alpha$

C: $\sin(60^\circ - \alpha) = \cos(30^\circ + \alpha)$ *(3 points)*

Solution: a) With the usual notation a, b, c for the sides of the triangle ABC and α, β, γ for its angles, the area of the triangle is

$$t = \frac{ab\sin\gamma}{2} = \frac{bc\sin\alpha}{2} = \frac{ac\sin\beta}{2}$$

The angle DAE is $360^\circ - 2\cdot 90^\circ - \alpha = 180^\circ - \alpha$, and $AD = c$, $AE = b$, so the area of the triangle ADE is

$$\frac{bc\sin(180^\circ - \alpha)}{2} = \frac{bc\sin\alpha}{2} = t$$

In the same way, the areas of the other two new triangles are also equal to the area of the original triangle.

b) We compute an angle of the triangle ABC by the law of cosines. With $a = 8$ cm, $b = 7$ cm, $c = 6$ cm,

$$64 = 49 + 36 - 2\cdot 7\cdot 6\cos\alpha, \qquad \text{so} \qquad \cos\alpha = \frac{1}{4} \quad \text{and} \quad \alpha \approx 75.52^\circ$$

So the area of the triangle is $T = \frac{7\cdot 6\cdot\sin\alpha}{2} = 21\cdot\frac{\sqrt{15}}{4} \approx 20.33$ cm^2. The hexagon consists of the triangle, the three squares, and the three triangles of part a), so its area is

$$a^2 + b^2 + c^2 + 4T = 64 + 49 + 36 + 4\cdot 20.33 \approx 230.33 \text{ cm}^2$$

c) A is true: $\cos(270^\circ + \alpha) = \sin\alpha$. B is false: $1 + \cos 2\alpha = 2\cos^2\alpha$, which is not always equal to $2\sin^2\alpha$ (for example, for $\alpha = 0$). C is true: $\sin(60^\circ - \alpha) = \cos(90^\circ - (60^\circ - \alpha)) = \cos(30^\circ + \alpha)$.

Note: In KoMaL's original, the points of Part I add up to 52.

Part II

5. The height of a triangle to its side b, its side a, its side b, and its side c, measured in centimeters, are four consecutive terms of an arithmetic sequence with common difference 2, in this order.

   a) How many times the numerical value of the perimeter is the numerical value of the area of the triangle? *(8 points)*

   b) Into what parts does the height to the side b divide the side b? *(3 points)*

   c) Prove that $2^{4n} + 4$ is divisible by 20 for every positive integer n. *(5 points)*

   Solution: a) Then $a = b - 2$, $c = b + 2$, and the height m to the side b is $m = b - 4$. Since $m > 0$, $b > 4$. We write the area of the triangle in two ways. First, with the side b and its height: $T = \frac{1}{2}bm = \frac{b(b-4)}{2}$. Second, by Heron's formula, $T = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{(b-2)+b+(b+2)}{2} = \frac{3b}{2}$ is half the perimeter. So

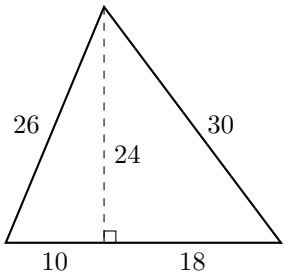
$$\sqrt{\frac{3b}{2}\cdot\frac{b+4}{2}\cdot\frac{b}{2}\cdot\frac{b-4}{2}} = \frac{b(b-4)}{2}$$

   Both sides are positive, so squaring is an equivalent transformation:

$$\frac{3b^2(b+4)(b-4)}{16} = \frac{b^2(b-4)^2}{4}$$

   Dividing by the positive $b^2(b-4)$ and multiplying by 16, we get $3(b+4) = 4(b-4)$, so $b = 28$ cm. Then $m = 24$ cm, and the other two sides are $a = 26$ cm and $c = 30$ cm. Such a triangle exists; its perimeter is $K = 84$ cm and its area is $T = \frac{28\cdot 24}{2} = 336$ cm^2. So the ratio of the area to the perimeter is $\frac{336}{84} = 4$.

   b) The height to the side b forms right triangles with the sides a and c and the two parts of b. By the Pythagorean theorem, the parts are $\sqrt{26^2 - 24^2} = 10$ cm and $\sqrt{30^2 - 24^2} = 18$ cm.



c) $2^{4n}+4=16^n+4$ is clearly divisible by 4. Every power of 16 ends in 6, so adding 4 gives a number ending in 0, which is divisible by 10. So $2^{4n}+4$ is divisible by the least common multiple of 4 and 10, which is 20.

The statement can also be proved by induction. For $n=1$: $2^4+4=20$ is divisible by 20. Suppose that $2^{4n}+4$ is divisible by 20. Then

$$2^{4(n+1)}+4=2^4\cdot 2^{4n}+2^4\cdot 4-2^4\cdot 4+4=2^4\left(2^{4n}+4\right)-60$$

which is divisible by 20, since $2^{4n}+4$ is divisible by 20 by the induction hypothesis, and 60 is a multiple of 20. So the statement holds for every positive integer n.

6. We draw two circles in the coordinate plane, with equations $x^2+y^2-14x-12y+60=0$ and $x^2+y^2-2x-6y=0$. Let O_1 be the center of the circle with the larger radius and O_2 the center of the circle with the smaller radius. Let A and B be the intersection points of the two circles.

a) Prove that the area of the triangle O_1AB is twice the area of the triangle O_2AB. *(8 points)*

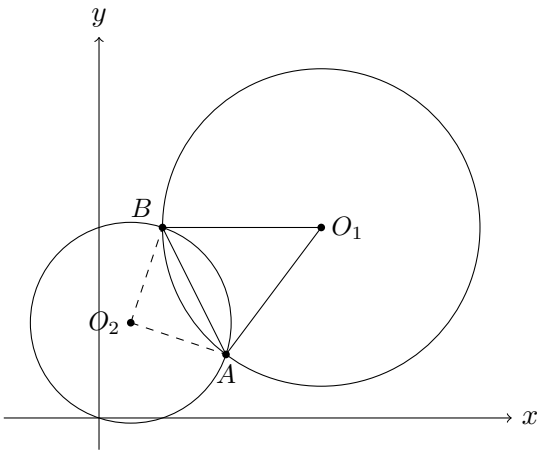
Consider the disks $x^2+y^2-14x-12y+60\le 0$ and $x^2+y^2-2x-6y\le 0$ as sets of points.

b) Find the area of the intersection of the two sets. *(5 points)*

c) Find the area of the union of the two sets. *(3 points)*

Solution: a) The first circle is $(x-7)^2+(y-6)^2=25$, with center $(7,6)$ and radius $r_1=5$. The second circle is $(x-1)^2+(y-3)^2=10$, with center $(1,3)$ and radius $r_2=\sqrt{10}$. The first radius is larger, so $O_1\,(7,6)$ and $O_2\,(1,3)$.

Subtracting the second equation from the first, we get $-12x-6y+60=0$, that is, $y=10-2x$. Substituting into the second equation and simplifying, $x^2-6x+8=0$, so $x_1=4$ and $x_2=2$, and then $y_1=2$ and $y_2=6$. So $A\,(4,2)$ and $B\,(2,6)$ (or the other way around).



The side BO_1 is parallel to the x-axis and has length 5, and the height to it from A is $6-2=4$, so the area of the triangle O_1AB is $t_1=\frac{5\cdot 4}{2}=10$.

The sides of the triangle O_2AB are $O_2A = O_2B = \sqrt{10}$ and $AB = \sqrt{20}$, so it is an isosceles right triangle, with area $\frac{\left(\sqrt{10}\right)^2}{2} = 5$. So the area of O_1AB is twice the area of O_2AB.

b) O_1 and O_2 are on opposite sides of the line AB, so the intersection of the two disks consists of two circular segments. The segment cut off from the smaller disk is the sector O_2AB (a quarter of the disk, since the angle at O_2 is $90°$) minus the triangle O_2AB:

$$\frac{\left(\sqrt{10}\right)^2 \pi}{4} - 5 \approx 2.85$$

For the segment of the larger disk, we compute the angle BO_1A by the law of cosines:

$$20 = 5^2 + 5^2 - 2 \cdot 5 \cdot 5 \cos\alpha, \qquad \cos\alpha = 0.6, \quad \alpha \approx 53.13°$$

The sector O_1AB minus the triangle O_1AB:

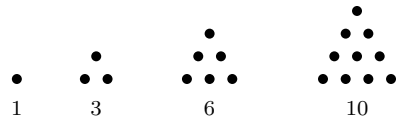
$$\frac{53.13°}{360°} \cdot 5^2\pi - 10 \approx 1.59$$

So the area of the intersection of the two disks is about 4.44 square units.

c) The area of the union is the sum of the areas of the two disks minus the area of their intersection:

$$5^2\pi + \left(\sqrt{10}\right)^2 \pi - 4.44 \approx 105.52 \text{ square units}$$

7. A number is called a triangular number if we can get it by adding all the natural numbers from 1 up to some number. The sum of the first n positive integers is the nth triangular number. The arrangement in the figure shows why they are called triangular numbers.



a) We choose one of the first hundred positive integers. Consider the following events:

A: the chosen number is a triangular number,

B: the chosen number is a square number,

C: the chosen number is a prime.

Find the following probabilities: $P(A)$, $P(A \cdot C)$, $P\left(\overline{A+B}\right)$. (Here $A \cdot C$ means "A and C", and $A + B$ means "A or B".) *(8 points)*

b) Adding the nth positive square number to the nth triangular number, we get $14\,751$. Find n and these numbers. *(8 points)*

Solution: a) The triangular numbers up to 100 are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, and 91. So there are 13 triangular numbers between 1 and 100, and $P(A) = 0.13$.

Only one of these is a prime (3), so $P(A \cdot C) = 0.01$.

Two of these are square numbers, 1 and 36. There are ten square numbers up to 100. So by inclusion-exclusion, there are $13 + 10 - 2 = 21$ numbers that are triangular numbers or square numbers:

$$P(A+B) = \frac{21}{100} = 0.21, \qquad \text{so} \qquad P\left(\overline{A+B}\right) = 1 - 0.21 = 0.79$$

b) The nth triangular number is the sum of the first n positive integers, $\frac{n(n+1)}{2}$. So

$$\frac{n\,(n+1)}{2}+n^2=14\,751, \qquad \text{that is,} \qquad 3n^2+n-29\,502=0$$

The positive solution is $n=99$. The 99th triangular number is 4950 and the 99th square number is 9801. Check: $4950+9801=14\,751$.

8. Consider the two functions defined on the set of positive real numbers $f:\mathbb{R}^+\to\mathbb{R}$, $f\,(x)=\cos x$ and $g:\mathbb{R}^+\to\mathbb{R}$, $g\,(x)=2x-\frac{\pi}{2}$.

a) Find the following values: $f\left(g\left(\frac{3\pi}{4}\right)\right)$, $g^{-1}\left(\frac{\pi}{2}\right)$, $g\left(f\left(\frac{\pi}{4}\right)\right)$. *(3 points)*

b) Find the area of the region enclosed by the y-axis, the graphs of f and g, and the line $x=\frac{\pi}{4}$. *(5 points)*

c) Solve the following equation on the interval $[0,2\pi]$.

$$\cos 2x+\sin\left(x+\frac{\pi}{3}\right)=\frac{1}{2}\sin x$$

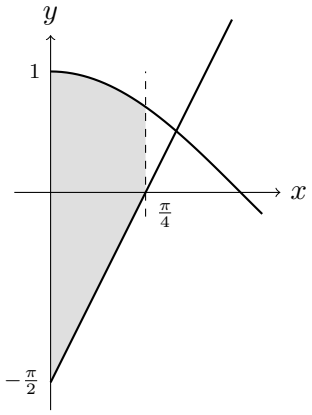
(8 points)

Solution: a) $f\left(g\left(\frac{3\pi}{4}\right)\right)=\cos\left(2\cdot\frac{3\pi}{4}-\frac{\pi}{2}\right)=\cos\pi=-1$.

The inverse of g is $g^{-1}\,(x)=\frac{x}{2}+\frac{\pi}{4}$, so $g^{-1}\left(\frac{\pi}{2}\right)=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$.

$g\left(f\left(\frac{\pi}{4}\right)\right)=2\cos\frac{\pi}{4}-\frac{\pi}{2}=\sqrt{2}-\frac{\pi}{2}$.

b) The region consists of two parts: the region under the graph of $\cos x$ on the interval $\left[0,\frac{\pi}{4}\right]$, and the triangle with vertices $(0,0)$, $\left(\frac{\pi}{4},0\right)$, $\left(0,-\frac{\pi}{2}\right)$.



$$\int_0^{\frac{\pi}{4}}\cos x\,dx=\left[\sin x\right]_0^{\frac{\pi}{4}}=\frac{\sqrt{2}}{2}\approx 0.7071$$

and the area of the triangle is $\frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{\pi}{4}=\frac{\pi^2}{16}\approx 0.6169$. So the area in question is about 1.324 square units.

c) Using $\cos 2x=\cos^2 x-\sin^2 x$ and the addition formula for sine,

$$\begin{aligned}\cos^2 x-\sin^2 x+\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3} &= \frac{1}{2}\sin x\\ \cos^2 x-\sin^2 x+\frac{\sqrt{3}}{2}\cos x &= 0\\ 2\cos^2 x+\frac{\sqrt{3}}{2}\cos x-1 &= 0\end{aligned}$$

So $\cos x=\frac{-\sqrt{3}\pm\sqrt{35}}{8}$, that is, $\cos x\approx 0.523$ or $\cos x\approx -0.956$. If $\cos x\approx 0.523$, then $x_1\approx 1.02$ and $x_2\approx 5.26$; if $\cos x\approx -0.956$, then $x_3\approx 2.84$ and $x_4\approx 3.44$. These four values are indeed solutions of the equation.

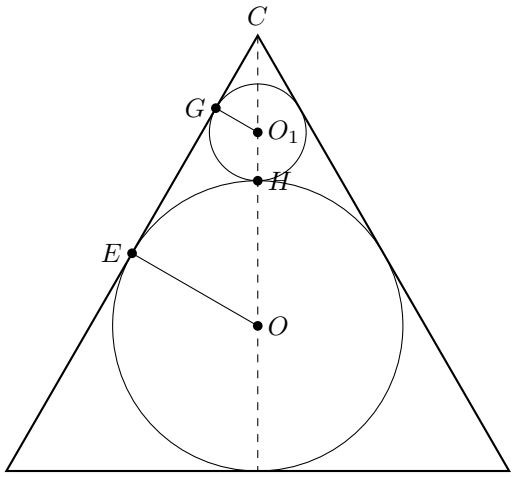
9. We place two balls one above the other in a right circular cone whose apex angle is $60°$, so that the larger ball rests on the base of the cone, the two balls touch each other, and both touch the lateral surface of the cone.

   a) Find the ratio of the radii of the two balls. *(7 points)*

   b) How many times the total volume of the two balls is the volume of the cone? *(6 points)*

   c) Find the volume of the cone, if its height is 18 cm. *(3 points)*

   Solution: a) We draw the cross-section of the cone through its axis. This triangle is equilateral, because it is symmetric and one of its angles is $60°$.



   The larger circle (the inscribed circle of the equilateral triangle) has center O and radius $R = OE$; the smaller circle has center O_1 and radius $r = O_1G$, where E and G are the points of tangency on a side, and H is the point where the two circles touch. The triangle CO_1G is similar to the triangle COE (their angles are equal). The angles of the triangle CO_1G are $30°$, $60°$, and $90°$, so $CO_1 = 2O_1G = 2r$; similarly, $CO = 2OE = 2R$. So $CH = CO_1 + O_1H = 3r$ and $CH = CO - OH = R$. So $R = 3r$, and the ratio of the radii is $1 : 3$.

   b) The height of the equilateral cross-section is $3R$ (the incenter divides the height in the ratio $1 : 2$). If the slant side of the cone is a, then $3R = a\frac{\sqrt{3}}{2}$, so $a = \frac{6R}{\sqrt{3}}$, and the radius of the base of the cone is $\frac{a}{2} = \frac{3R}{\sqrt{3}} = R\sqrt{3}$. The volume of the cone is

$$V = \frac{\left(R\sqrt{3}\right)^2 \pi \cdot 3R}{3} = 3\pi R^3 = 81\pi r^3$$

   The total volume of the balls is

$$V' = \frac{4\pi r^3}{3} + \frac{4\pi R^3}{3} = \frac{4\pi r^3}{3} + \frac{4 \cdot 27\pi r^3}{3} = \frac{112}{3}\pi r^3$$

   So the volume of the cone is $\frac{81 \cdot 3}{112} = \frac{243}{112} \approx 2.17$ times the total volume of the balls.

   c) The height of the cone is $m = 18$ cm $= 3R$, so $R = 6$ cm. The volume of the cone is

$$V = 3\pi R^3 = 648\pi \approx 2035.75 \text{ cm}^3$$

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