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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.
The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. a) Solve the following equation over the set of integers.

$$\left(x^2-9\right)\left(\frac{1}{x-3}-\frac{1}{x+3}-1\right)=9+x$$

(6 points)

b) An angle α of a quadrilateral satisfies $4\sin^2\alpha-3=0$. What can the measure of the angle α be? *(6 points)*

Solution: a) The domain of the equation is $x\neq\pm3$. Since $x^2-9=(x-3)(x+3)$, multiplying out we get

$$(x+3)-(x-3)-\left(x^2-9\right)=9+x$$

which leads to the quadratic equation $x^2+x-6=0$. Its roots are $x_1=-3$ and $x_2=2$, but x_1 is not in the domain. Since we used equivalent transformations, the solution of the equation is $x=2$.

b) $\sin^2\alpha=\frac{3}{4}$, so $\sin\alpha=\frac{\sqrt{3}}{2}$ or $\sin\alpha=-\frac{\sqrt{3}}{2}$. An angle of a quadrilateral is between 0° and 360° (a non-convex quadrilateral has an angle greater than 180°). In this interval, the solutions of $\sin\alpha=\frac{\sqrt{3}}{2}$ are $\alpha_1=60^\circ$ and $\alpha_2=120^\circ$, and the solutions of $\sin\alpha=-\frac{\sqrt{3}}{2}$ are $\alpha_3=240^\circ$ and $\alpha_4=300^\circ$. All four values satisfy the equation.

Note: The points for part b) are missing from KoMaL's text; for the total of 51 *points of Part I, they must be* 6.

2. The wheels of Zoli's bike have a diameter of 70 cm. The front gear, attached to the pedals, is set to a gear wheel with 56 "teeth" around its circumference, and the rear gear, attached to the rear wheel, is set to a gear wheel with 20 "teeth". (The front gear wheel turns together with the pedals. The bike chain connects the front and rear gear wheels, so the two gear wheels always turn by the same number of "teeth".)

a) How fast does Zoli ride, if the pedals make 10 full turns in 8 seconds, and the rear wheel turns together with the rear gear wheel? *(4 points)*

b) Zoli goes on a bike tour with 34 friends. Each wheel of each bike, independently of the others, gets a flat tire between two whole kilometers with probability 0.0005. For what length of the tour can they say that there will be a flat tire on the tour with probability at least 0.95? *(6 points)*

c) On Tuesday, 48 people rented a bike at a bike rental shop, 4 more women than men. The number of guests younger than 40 was 60% of the number of guests aged at least 40. Among those aged at least 40, five times as many asked for a regular bike as for an electric bike. Exactly one man asked for an electric bike, and there were 2 more women than men aged at least 40. How many women younger than 40 rented a bike on Tuesday? *(4 points)*

Solution: a) In 8 seconds the front gear wheel turns by $10\cdot56=560$ "teeth". The rear gear wheel turns by the same number of teeth, which is $560:20=28$ full turns of the rear wheel. The circumference of the wheel is $2r\pi=70\pi$ cm, so Zoli covers $28\cdot70\pi\approx6158$ cm ≈61.58 m in 8 seconds. So his speed is $61.58:8\approx7.7$ m/s.

b) The probability that a wheel does not get a flat tire in n km is 0.9995^n. The wheels get flat tires independently, so the probability that none of the 70 wheels gets a flat tire in n km is $(0.9995^n)^{70} = 0.9995^{70n}$. The complement, that there is a flat tire in n km, has probability $1 - 0.9995^{70n}$, and we need

$$1 - 0.9995^{70n} \geq 0.95, \qquad \text{that is,} \qquad 0.05 \geq 0.9995^{70n}$$

The function 0.9995^x is strictly decreasing, so $70n \geq \log_{0.9995} 0.05$, that is,

$$n \geq \frac{\log_{0.9995} 0.05}{70} \approx 85.6$$

So for a tour of at least 86 km, the probability of a flat tire is at least 0.95.

c) $\frac{48-4}{2} = 22$ men and 26 women rented a bike. If y guests were at least 40 years old, then $y + 0.6y = 48$, so $y = \frac{48}{1.6} = 30$ guests were at least 40, and 18 were younger. Of those aged at least 40, $30 \cdot \frac{5}{6} = 25$ asked for a regular bike and 5 for an electric bike. Let x be the number of women younger than 40:

	women	men	total
younger than 40	x	$18 - x$	18
at least 40, electric bike	4	1	5
at least 40, regular bike	$22 - x$	$x + 3$	25
total	26	22	48

By the condition on the women and men aged at least 40,

$$4 + (22 - x) = 1 + (x + 3) + 2$$

so $x = 10$: 10 women younger than 40 rented a bike on Tuesday.

3. a) Find the values of the real parameter m for which the quadratic equation $(m-1)\,x^2 + (2m-9)\,x + 1 = 0$ has two different positive roots whose sum is greater than 5. *(7 points)*

b) I thought of a number, wrote it on a sheet of paper, and wrote its square next to it. Can the number I thought of be even, if the sum of the base 10 logarithms of the numbers written down is less than the base 10 logarithm of the sum of the numbers? *(6 points)*

Solution: a) $m \neq 1$, since otherwise the equation would not be quadratic. A quadratic equation has two different real roots exactly when its discriminant is positive:

$$(2m-9)^2 - 4\,(m-1) > 0, \qquad \text{that is,} \qquad 4m^2 - 40m + 85 > 0$$

which holds for $m < \frac{10-\sqrt{15}}{2} \approx 3.06$ or $m > \frac{10+\sqrt{15}}{2} \approx 6.94$. By Viète's formulas, for the roots x_1 and x_2,

$$x_1 x_2 = \frac{1}{m-1} > 0$$

so $m > 1$. Also,

$$x_1 + x_2 = -\frac{2m-9}{m-1} > 5$$

The denominator is positive by the previous condition, so multiplying by it, $-(2m-9) > 5\,(m-1)$, that is, $m < 2$. (Then the sum of the roots is positive, so both roots are positive.) Combining the conditions, the values $1 < m < 2$ satisfy the requirements.

Note: KoMaL's solution writes the product of the roots as $\frac{25}{m-1}$; it is $\frac{1}{m-1}$. The conclusion $m > 1$ is the same.

b) Let x be the number; because of the logarithm, $x > 0$. We solve the inequality $\lg x + \lg x^2 < \lg\left(x + x^2\right)$. By the product rule of logarithms, $\lg x^3 < \lg\left(x + x^2\right)$. The base 10 logarithm is strictly increasing, so $x^3 < x + x^2$, and dividing by $x > 0$, we get $x^2 - x - 1 < 0$. Its solution is the interval $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2} \approx 1.618$, which contains no positive even number. So the number cannot be even.

4. a) The sum of the first three terms of an arithmetic sequence with positive terms is 10.5. The sum of the values of the function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = 2^x$ at the first three terms of the sequence is $73\sqrt{2}$. Find the common difference of the sequence. *(7 points)*

   b) Prove that if the sum of the first n terms of a sequence is $S_n = 1.5n^2 - n$, then it is an arithmetic sequence with common difference 3. *(5 points)*

   Solution: a) Let a_2 be the second term and d the common difference. Then $(a_2 - d) + a_2 + (a_2 + d) = 10.5$, so $a_2 = 3.5$. Since $2^{3.5} = 8\sqrt{2}$, the sum of the values of the function is

$$2^{3.5-d} + 2^{3.5} + 2^{3.5+d} = 73\sqrt{2}, \qquad \text{so} \qquad \frac{8\sqrt{2}}{2^d} + 8\sqrt{2} \cdot 2^d = 65\sqrt{2}$$

   With the new variable $y = 2^d$, and dividing by $\sqrt{2}$, we get the equation $8y^2 - 65y + 8 = 0$. Its solutions are $y_1 = 8$, so (since 2^x is strictly monotonic) $d_1 = 3$, and $y_2 = \frac{1}{8}$, so $d_2 = -3$. Both values satisfy the conditions: the first three terms are 0.5, 3.5, 6.5 or 6.5, 3.5, 0.5.

   b) The nth term is $a_n = S_n - S_{n-1}$ for $n \geq 2$:

$$a_n = \left(1.5n^2 - n\right) - \left(1.5\,(n-1)^2 - (n-1)\right) = 3n - 2.5$$

   This formula also holds for $n = 1$, since $a_1 = S_1 = 0.5$. So $a_n - a_{n-1} = (3n - 2.5) - (3\,(n-1) - 2.5) = 3$, and the sequence is arithmetic with common difference 3.

Part II

5. In the park of a castle, the tulip bed has the shape of a right trapezoid with parallel sides 8 m and 5 m, and the leg perpendicular to them is 4 m long.

   a) Prove that the tulips planted along the diagonal from the vertex with the acute angle are on the bisector of the acute angle. *(6 points)*

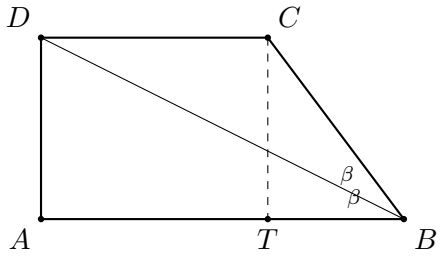
   The gardeners sow grass on an area in the shape of an isosceles trapezoid with bases 6 m and 3 m, whose acute angles are $60°$. Children throw plastic hula hoops of radius 30 cm onto this area. The (imaginary) centers of the hoops fall at random onto the grass area.

   b) What is the probability that a hula hoop does not hang over the edge of the grass area? *(5 points)*

   In the pond in the middle of the park there are water lilies. During the blooming season, every day we can see 30% more open flowers than the day before.

   c) If we counted twelve flowers on June 1, when will there be at least two hundred open flowers on the pond? *(5 points)*

   Solution: a) Let the trapezoid be $ABCD$ with $AB = 8$, $CD = 5$, $AD = 4$, and right angles at A and D; the acute angle α is at B. Let T be the foot of the perpendicular from C to AB.



   In the right triangle TBC, $TB = 8 - 5 = 3$ m, so $\tan\alpha = \frac{4}{3}$. In the right triangle ABD, the angle $\beta = \angle ABD$ satisfies $\tan\beta = \frac{4}{8} = \frac{1}{2}$. By the double angle formula,

$$\tan 2\beta = \frac{2\tan\beta}{1 - \tan^2\beta} = \frac{2 \cdot \frac{1}{2}}{1 - \frac{1}{4}} = \frac{4}{3}$$

The tangent function is one-to-one on acute angles, so $2\beta = \alpha$, that is, the diagonal BD bisects the angle at B.

b) The hoop does not hang over the edge exactly when its center is at least 0.3 m from every side of the trapezoid $EFGH$, that is, inside the trapezoid $PQRS$ whose sides are parallel to those of $EFGH$, 0.3 m inside it.

The legs of $EFGH$ are 3 m (since $\frac{6-3}{2} = 1.5 = 3\cos 60°$), so its height is $m_1 = \frac{3\sqrt{3}}{2}$ and its area is $t_1 = \frac{6+3}{2} \cdot \frac{3\sqrt{3}}{2} = \frac{27\sqrt{3}}{4} \approx 11.69$ m^2. At the $60°$ angles the inner corners are $0.3 \cdot \cot 30° = \frac{3\sqrt{3}}{10}$ m in from the corners along the long base, and at the $120°$ angles they are $0.3 \cdot \cot 60° = \frac{\sqrt{3}}{10}$ m in along the short base. So the parallel sides of $PQRS$ are

$$PQ = 6 - 2 \cdot \frac{3\sqrt{3}}{10} \approx 4.96 \text{ m} \qquad \text{and} \qquad RS = 3 - 2 \cdot \frac{\sqrt{3}}{10} \approx 2.65 \text{ m}$$

and its height is $m_2 = \frac{3\sqrt{3}}{2} - 2 \cdot 0.3 \approx 2.00$ m, so its area is $t_2 \approx 7.61$ m^2. By the geometric probability model, the probability is the ratio of the two areas: $P = \frac{t_2}{t_1} \approx 0.651$.

c) The numbers of open flowers form a geometric sequence with first term $a_1 = 12$ and ratio $q = 1.3$. We solve the inequality $a_n = 12 \cdot 1.3^{n-1} \geq 200$: $n - 1 \geq \log_{1.3} \frac{200}{12} \approx 10.7$, so $n \geq 11.7$. So from June 12 on there will be at least 200 open flowers on the pond.

6. The set H consists of the pairs of numbers (a, b) that satisfy the following conditions: a and b are natural numbers, $a \leq b$, and $a \mid 12$, $b \mid 12$.

a) Prove that H has 21 elements. *(3 points)*

Consider the graph with 21 vertices, the elements of H. In this graph, we connect two vertices if the corresponding pairs have a number in common. For example, we connect $(4, 6)$ and $(6, 12)$, because both pairs contain 6.

b) Give a cycle of 5 vertices in this graph by listing its vertices. *(2 points)*

c) How many edges does this graph have? *(5 points)*

d) We want to draw a regular 21-gon, corresponding to the 21 vertices of the graph, on a circular disk, so that the longest diagonal of the regular polygon is 18 cm. At least what radius does the disk need? *(6 points)*

Solution: a) The positive divisors of 12 are 1, 2, 3, 4, 6, 12; a and b are chosen from these, with repetition allowed. If $a = 1$, then b can take 6 values; if $a = 2$, then b can take 5 values, and so on. So the number of pairs is $6 + 5 + 4 + 3 + 2 + 1 = 21$.

Remark: Using the formula for combinations with repetition, the number is $\binom{6+2-1}{2} = \binom{7}{2} = 21$.

b) For example, $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 6)$, $(1, 6)$.

c) Each of the 6 vertices with two equal numbers, (a, a), is connected to the 5 other pairs containing a. Each of the 15 vertices with two different numbers, (a, b), is connected to the 5 other pairs containing a and the 5 other pairs containing b, so to 10 vertices. The sum of the degrees is $6 \cdot 5 + 15 \cdot 10 = 180$, so the number of edges is 90.

d) An interior angle of the regular 21-gon is $\frac{19 \cdot 180°}{21} \approx 162.86°$, and two neighboring diagonals from a vertex form an angle $\alpha = \frac{180°}{21} \approx 8.57°$. The two longest diagonals from a vertex (18 cm each) also form this angle, and the opposite side of this isosceles triangle is a side of the polygon, so the side of the polygon is $a = 2 \cdot 18 \cdot \sin \frac{\alpha}{2} \approx 2.69$ cm. The radius of the circumscribed circle of the polygon is

$$R = \frac{a}{2 \sin \alpha} \approx 9.03 \text{ cm}$$

(Or directly: the longest diagonal is the chord over 10 sides, with central angle $\frac{10 \cdot 360°}{21}$, so $18 = 2R \sin \frac{10 \cdot 180°}{21}$ and $R = \frac{9}{\sin 85.71°} \approx 9.03$ cm.) So the disk needs a radius of at least about 9.03 cm.

7. a) In Szilárd's weather forecast, the daily highs from Monday to Friday are 34°C, 30°C, 26°C, 22°C, and 28°C. Szilárd also said that for the whole week the mean of the daily highs is 28°C, and the mean absolute deviation from the mean is $\frac{18}{7}$°C. Find the daily highs on the weekend, if it is expected to cool down a little on Sunday. *(6 points)*

b) For the year 2023, Szilárd collected the daily precipitation, and gave the data needed to draw the box plot: $Q_0 = 0$ mm, $Q_1 = 1$ mm, $Q_2 = 5$ mm, $Q_3 = 8$ mm, and $Q_4 = 32$ mm. What can the smallest and the largest possible total precipitation be in 2023? *(5 points)*

c) The digit 7 on Szilárd's computer broke, so he uses base 7 instead of base 10 to send data. What are the wind speed and the highest and lowest temperatures in base 10, if Szilárd's message was this:

"The highest wind speed measured today is 50 km/h. The sum of the highest and lowest temperatures is 105°C, and their difference is 15°C." *(5 points)*

Solution: a) Let x_1 be the high on Saturday and x_2 on Sunday, with $x_1 > x_2$. The deviations of the weekday highs from 28 are 6, 2, 2, 6, 0, with sum 16. We solve the system

$$\frac{140 + x_1 + x_2}{7} = 28, \qquad \frac{16 + |x_1 - 28| + |x_2 - 28|}{7} = \frac{18}{7}$$

From the first equation $x_1 + x_2 = 56$, so $x_1 = 28 + t$ and $x_2 = 28 - t$ for some $t > 0$; the second equation gives $2t = 2$, $t = 1$. So the high on Saturday is 29°C and on Sunday 27°C.

b) With the 365 data values in increasing order, the first one is the minimum, the lower quartile is the mean of the 91st and 92nd values, the median is the 183rd value, the upper quartile is the mean of the 274th and 275th values, and the 365th value is the maximum. So the smallest possible total precipitation is

$$90 \cdot 0 + 92 \cdot 1 + 91 \cdot 5 + 91 \cdot 8 + 32 = 1307 \text{ mm}$$

and the largest possible total is

$$1 \cdot 0 + 91 \cdot 1 + 91 \cdot 5 + 92 \cdot 8 + 90 \cdot 32 = 4162 \text{ mm}$$

c) $50_7 = 5 \cdot 7 = 35$, so the highest wind speed is 35 km/h. In base 7, the highest temperature is $\frac{105_7 + 15_7}{2}$: since $105_7 = 54$ and $15_7 = 12$, the highest temperature is $\frac{54+12}{2} = 33$°C (45 in base 7) and the lowest is $\frac{54-12}{2} = 21$°C (30 in base 7).

8. a) We put a regular square pyramid of height 6 cm on top of a square prism, so that the base of the pyramid fits exactly on the square face of the prism. Find the surface area of the solid, if its base edges are 12 cm long and its total height is 30 cm. *(4 points)*

b) We write positive primes on the edges of the solid above, so that the numbers on the edges meeting at any vertex are different. Is it possible that the sum of the numbers written on the edges is less than 70? Justify your answer. *(4 points)*

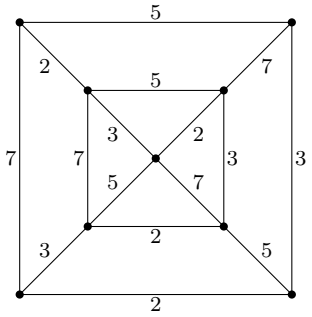
c) We have a square prism of height 22 cm, with a regular square pyramid of height 6 cm attached to one of its square faces. We stand the solid on its square face, pour 500 ml of liquid into it, and mark the liquid level. Then we stand the solid on the apex of the pyramid, and find that the liquid level is now 2 cm higher than the mark, but both levels are in the prism part. How long are the base edges of the prism? *(8 points)*

Solution: a) The height of the prism is $30 - 6 = 24$ cm. The surface area of the prism without its top face is

$$A_1 = 12 \cdot 12 + 4 \cdot 12 \cdot 24 = 1296 \text{ cm}^2$$

The height of a lateral face of the pyramid is $m = \sqrt{6^2 + 6^2} = 6\sqrt{2}$ cm, so the four lateral faces together have area $A_2 = 4 \cdot \frac{12 \cdot 6\sqrt{2}}{2} = 144\sqrt{2} \approx 203.65$ cm^2. So the surface area of the solid is $A \approx 1500$ cm^2.

b) Yes, it is possible. We draw the edges of the solid as a graph (the apex in the middle, the top square of the prism around it, and the bottom square outside), and write small primes on the edges so that the edges at each vertex get different numbers:



The sum of the primes on the edges is $68 < 70$. (The four edges at the apex need four different primes, and a computer search confirms that 68 is the smallest possible sum.)

c) 500 ml $= 500$ cm^3. Let the base edge of the prism be a cm. Standing on the square face, the liquid fills the prism up to height m_1:

$$a^2 m_1 = 500$$

Standing on the apex, the liquid fills the whole pyramid and a prism-shaped part of height m_2 above it:

$$\frac{a^2 \cdot 6}{3} + a^2 m_2 = 500$$

In the first position the mark is $22 - m_1$ cm below the top of the prism part, which is $6 + (22 - m_1)$ cm above the apex in the second position; the new level is $6 + m_2$ cm above the apex. Because of the 2 cm difference, $m_1 + m_2 = 22 + 2 = 24$. Expressing m_1 and m_2 from the first two equations and substituting,

$$\frac{500}{a^2} + \frac{500 - 2a^2}{a^2} = 24, \qquad \text{so} \qquad \frac{1000}{a^2} = 26$$

and the positive solution is $a = \sqrt{\frac{500}{13}} \approx 6.2$ cm. (Then $m_1 = 13$ cm and $m_2 = 11$ cm, both levels in the prism part.) So the base edges of the prism are about 6.2 cm long.

9. In the oral part of the advanced-level school-leaving exam in mathematics, an important element is presenting practical applications of the given topic. In connection with differential and integral calculus, students often bring examples from physics.

a) The speed of a car as a function of time is a quadratic function whose two zeros are at 0 seconds and 10 seconds, and whose maximum value, at the 5th second of the motion, is 8 m/s. Prove that the rule of the speed function is $f(x) = -0.32x^2 + 3.2x$. *(4 points)*

b) The distance traveled by the car is the area under the speed–time graph. Find the distance traveled by the car until it stops. *(4 points)*

c) The acceleration of a body can be computed from its speed–time function as the slope of the tangent line at the given time. For a body in simple harmonic motion, the speed is given by $f(x) = 0.3\cos(3x)$. Compute the acceleration at $x = 2$ seconds. *(4 points)*

d) The work done is the product of the force and the displacement in the direction of the force. How much work is done if a constant force of 45 N acts on a body in the positive direction of the x-axis, while the body travels 160 m up the slope along the graph of $f(x) = \frac{1}{2}x$, starting from the origin? *(4 points)*

Solution: a) Let the quadratic speed function be $f(x) = ax^2 + bx + c$. It passes through the origin, so $c = 0$. The other zero is $x = 10$, so $100a + 10b = 0$, and from the maximum, $f(5) = 25a + 5b = 8$ (or $-\frac{b}{2a} = 5$). From these, $a = -0.32$ and $b = 3.2$, so the rule is indeed $f(x) = -0.32x^2 + 3.2x$.

b) The area under the graph is

$$\int_0^{10} \left(-0.32x^2 + 3.2x\right) dx = \left[-0.32 \cdot \frac{x^3}{3} + 3.2 \cdot \frac{x^2}{2}\right]_0^{10} \approx 53.33$$

so the car travels about 53.33 meters.

c) The slope of the tangent line is given by the derivative: $f'(x) = -0.3 \sin(3x) \cdot 3$. So the acceleration at $x = 2$ is

$$f'(2) = -0.9 \sin 6 \approx 0.25$$

d) The slope of the line $f(x) = \frac{1}{2}x$ is $\frac{1}{2}$, so its angle of inclination is $\alpha = \tan^{-1} \frac{1}{2} \approx 26.6°$. The displacement in the direction of the force is

$$160 \cos \alpha = 160 \cdot \frac{2}{\sqrt{5}} \approx 143.11 \text{ m}$$

so the work done is $W \approx 45 \cdot 143.11 \approx 6440$ J.

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