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For high school students preparing for universities and colleges, one of the greatest challenges is the entrance examination. KoMaL wants to help them prepare by publishing problem sets that are similar in difficulty to the written entrance examination. This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time.

1. Solve the equation $x^2 - x = a^2 - \frac{1}{4}$, where a is a real parameter.

Solution: We complete the square on the left-hand side.

$$\begin{aligned} x^2 - x &= a^2 - \frac{1}{4} && \text{add } \frac{1}{4} \\ x^2 - x + \frac{1}{4} &= a^2 && \text{factor} \\ \left(x - \frac{1}{2}\right)^2 &= a^2 \\ x - \frac{1}{2} &= \pm a && \text{add } \frac{1}{2} \\ x &= \frac{1}{2} \pm a \end{aligned}$$

So the solutions are $x_1 = \frac{1}{2} + a$ and $x_2 = \frac{1}{2} - a$. If $a = 0$, these are the same, and the only solution is $x = \frac{1}{2}$.

2. The legs a and b and the hypotenuse c of a right triangle satisfy $(a - b)^2 = c^2 - b^2$. Find the altitude belonging to the hypotenuse.

Solution: By the Pythagorean theorem, $c^2 = a^2 + b^2$, and so $c^2 - b^2 = a^2$. The condition becomes $(a - b)^2 = a^2$, and so $a - b = a$ or $a - b = -a$. The first one gives $b = 0$, which is impossible. So $b = 2a$, and then

$$c = \sqrt{a^2 + b^2} = \sqrt{a^2 + 4a^2} = \sqrt{5}a$$

We compute the area of the triangle in two different ways: using the two legs, and using the hypotenuse and its altitude m .

$$\frac{ab}{2} = \frac{cm}{2} \quad \text{and so} \quad m = \frac{ab}{c} = \frac{a \cdot 2a}{\sqrt{5}a} = \frac{2a}{\sqrt{5}} = \frac{2\sqrt{5}}{5}a$$

Since $c = \sqrt{5}a$, we can also write this as $m = \frac{2}{5}c$. So the altitude belonging to the hypotenuse is two fifths of the hypotenuse.

3. The line $3x + 4y = 12$ intersects the coordinate axes at points A and B . The perpendicular bisector of segment AB splits triangle AOB into two parts. Find the areas of these parts. (O is the origin.)

Solution: If $y = 0$ then $x = 4$, and if $x = 0$ then $y = 3$. So $A(4, 0)$ and $B(0, 3)$. Triangle AOB is a right triangle with legs 4 and 3, so its area is 6 and its hypotenuse is $AB = 5$.

Let F denote the midpoint of AB , so $F\left(2, \frac{3}{2}\right)$ and $AF = \frac{5}{2}$. The slope of line AB is $-\frac{3}{4}$, so the slope of the perpendicular bisector is $\frac{4}{3}$, and its equation is

$$y - \frac{3}{2} = \frac{4}{3}(x - 2)$$

With $y = 0$ we get $x = \frac{7}{8}$, and with $x = 0$ we get $y = -\frac{7}{6}$. So the perpendicular bisector intersects side OA at $P\left(\frac{7}{8}, 0\right)$, and it does not intersect side OB . The line cuts triangle AOB into triangle APF and quadrilateral $OPFB$.

Triangle APF has a right angle at F and shares the angle at A with triangle AOB , which has a right angle at O . So the two triangles are similar, and the ratio of similarity is

$$\frac{AF}{AO} = \frac{\frac{5}{2}}{4} = \frac{5}{8}$$

The ratio of the areas of similar triangles is the square of the ratio of similarity. So the area of triangle APF is

$$6 \cdot \left(\frac{5}{8}\right)^2 = 6 \cdot \frac{25}{64} = \frac{75}{32}$$

and the area of quadrilateral $OPFB$ is $6 - \frac{75}{32} = \frac{117}{32}$. So the two parts have areas $\frac{75}{32}$ and $\frac{117}{32}$ (their ratio is 25 : 39).

4. Solve the given equation.

$$\log_{\sin x} (1 - \cos^2 x) = 2^{\sqrt{4-x^2}}$$

Solution: Consider the domain of this equation first. The base of a logarithm must be positive and not 1, so $0 < \sin x < 1$. The expression under the square root must be non-negative, so $4 - x^2 \geq 0$, and so $-2 \leq x \leq 2$.

Since $1 - \cos^2 x = \sin^2 x$, the left-hand side is

$$\log_{\sin x} (1 - \cos^2 x) = \log_{\sin x} (\sin^2 x) = 2$$

for every x in the domain. So the equation becomes

$$\begin{aligned} 2^{\sqrt{4-x^2}} &= 2 = 2^1 && \text{the exponential function is one-to-one} \\ \sqrt{4-x^2} &= 1 && \text{square both sides} \\ 4-x^2 &= 1 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

Both numbers are in the interval $[-2, 2]$. Since $\sqrt{3} \approx 1.73$ is between 0 and π , $\sin \sqrt{3} > 0$; and since $\sqrt{3} \neq \frac{\pi}{2}$, $\sin \sqrt{3} \neq 1$. So $x = \sqrt{3}$ is a solution. But $\sin(-\sqrt{3}) = -\sin \sqrt{3} < 0$, so $-\sqrt{3}$ is not in the domain. The only solution is $x = \sqrt{3}$.

5. For what values of the parameter α does the equation $\sin^4 x + \cos^4 x = \sin^2 \alpha$ have a solution?

Solution: We rewrite the left-hand side.

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2} (2\sin x \cos x)^2 = 1 - \frac{1}{2} \sin^2 2x$$

Since $0 \leq \sin^2 2x \leq 1$, the left-hand side takes on every value between $\frac{1}{2}$ and 1, and no other value. So the equation has a solution exactly when

$$\frac{1}{2} \leq \sin^2 \alpha \leq 1 \quad \text{or} \quad |\sin \alpha| \geq \frac{\sqrt{2}}{2}$$

This happens when

$$45^\circ + k \cdot 180^\circ \leq \alpha \leq 135^\circ + k \cdot 180^\circ \quad \text{or, in radians,} \quad \frac{\pi}{4} + k\pi \leq \alpha \leq \frac{3\pi}{4} + k\pi$$

where k is any integer.

6. One of the acute angles of a right triangle is 30° and its hypotenuse is 4 units long. Find the area of the triangle whose vertices are the points where the inscribed circle is tangent to the sides.

Solution: The side opposite the 30° angle is half of the hypotenuse, so the legs are 2 and $\sqrt{4^2 - 2^2} = 2\sqrt{3}$. In a right triangle, the radius of the inscribed circle is

$$r = \frac{a + b - c}{2} = \frac{2 + 2\sqrt{3} - 4}{2} = \sqrt{3} - 1$$

(The tangent segments from each vertex are equal, and the two tangent segments from the right angle vertex are both r .)

Let I denote the center of the inscribed circle. The radii drawn to the points of tangency are perpendicular to the sides. So at a vertex with angle γ , the radii drawn to the two points of tangency on the sides of that angle form an angle $180^\circ - \gamma$ at I . The angles of our triangle are 90° , 60° , and 30° , so the angles at I are 90° , 120° , and 150° . The triangle formed by the three points of tangency is split by the three radii into three isosceles triangles, each with two sides r . So its area is

$$\begin{aligned} T &= \frac{1}{2} r^2 (\sin 90^\circ + \sin 120^\circ + \sin 150^\circ) = \frac{1}{2} r^2 \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2} \right) = \frac{3 + \sqrt{3}}{4} r^2 \\ &= \frac{3 + \sqrt{3}}{4} (\sqrt{3} - 1)^2 = \frac{(3 + \sqrt{3})(4 - 2\sqrt{3})}{4} = \frac{12 - 6\sqrt{3} + 4\sqrt{3} - 6}{4} = \frac{6 - 2\sqrt{3}}{4} = \frac{3 - \sqrt{3}}{2} \end{aligned}$$

So the area is $\frac{3 - \sqrt{3}}{2} \approx 0.634$ square units.

7. Find all integers p and q that satisfy the equation $p^3 + 6p^2 + 5p = 3q^2 + 3q + 1$.

Solution: We factor the left-hand side.

$$p^3 + 6p^2 + 5p = p(p^2 + 6p + 5) = p(p + 1)(p + 5)$$

Since $p + 5 = (p + 2) + 3$, the factor $p + 5$ gives the same remainder when divided by 3 as $p + 2$. Among the three consecutive integers p , $p + 1$, and $p + 2$, one is divisible by 3. So one of the factors p , $p + 1$, and $p + 5$ is divisible by 3, and so the left-hand side is divisible by 3 for every integer p .

The right-hand side is $3q^2 + 3q + 1 = 3(q^2 + q) + 1$, which gives a remainder of 1 when divided by 3, for every integer q . So the two sides can never be equal: there are no integers p and q that satisfy the equation.

8. Find all pairs of numbers (x, y) that satisfy the given equation.

$$\sqrt{x + y - \sqrt{x + y - 1}} + \sqrt{x^2 - x + 1} = \frac{\sqrt{3}}{2}$$

Solution: For the first square root, $x + y - 1 \geq 0$. Let $t = \sqrt{x + y - 1}$, so $t \geq 0$ and $x + y = t^2 + 1$. We complete the square under both square roots.

$$\begin{aligned} x + y - \sqrt{x + y - 1} &= t^2 - t + 1 = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} \\ x^2 - x + 1 &= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} \end{aligned}$$

So each square root on the left-hand side is at least $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$, and their sum is at least $\sqrt{3}$. But the right-hand side is only $\frac{\sqrt{3}}{2} < \sqrt{3}$. So the two sides can never be equal, and there is no pair (x, y) that satisfies the equation.